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THE ASYMPTOTIC SHIFT FOR THE PRINCIPAL EIGENVALUE FOR SECOND ORDER ELLIPTIC OPERATORS IN THE PRESENCE OF SMALL OBSTACLES

BY

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ABSTRACT

Let L be a uniformly elliptic second order differential operator with nice coefficients, defined on a smooth, bounded domain in $\mathbb{R}^d$, $d \geq 2$, with either the Dirichlet or an oblique-derivative boundary condition. In this work we study the asymptotics for the principal eigenvalue of L under hard and soft obstacle perturbations. The hard obstacle perturbation of L is obtained by making a finite number of holes with the Dirichlet boundary condition on their boundaries. The main result gives the asymptotic shift of the principal eigenvalue as the holes shrink to points. The rates are expressed in terms of the Newtonian capacity of the holes and the principal eigenfunctions for the unperturbed operator and its formal adjoint. The soft obstacle corresponds to a finite number of compactly supported finite potential wells. Here we only consider the oblique-derivative Laplacian. The main difference from the hard obstacle problem is that phase transitions occur, due to the various scaling possibilities. Our results generalize known results on similar perturbations for selfadjoint operators. Our approach is probabilistic.

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1. Introduction and statement of results

Let $D \subset\subset \mathbb{R}^d,\ d \geq 2$, be a smooth, bounded domain. Define the second order elliptic operator $\mathrm{L} = \mathrm{L}_0 + V$, where

$$\mathrm{L}_0 = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla,$$

$a = \{a_{i,j}\}_{i,j=1}^d \in C^{2,\alpha}(\mathbb{R}^d)$ is uniformly elliptic on $\mathbb{R}^d$, $b = (b_1, \ldots, b_d) \in C^{1,\alpha}(\mathbb{R}^d)$ and $V \in C^\alpha(\mathbb{R}^d)$, for some $\alpha \in (0, 1)$. The boundary condition on ∂D which we relate to L is one of the following:

BC 1. *Dirichlet boundary condition.*

BC 2. *ν-oblique derivative boundary condition.* Let $\nu : \partial D \to S^{d-1}$ be smooth and satisfy $\nu(x) \cdot \overrightarrow{n}(x) > 0$, where $\overrightarrow{n}$ is the inward unit normal to D at ∂D. Then u satisfies the ν-oblique derivative boundary condition if $\nabla u \cdot \nu = 0$ on ∂D.

The first main result is the asymptotic behavior of the shift in the principal eigenvalue of L when a small set, A_ϵ, is removed from D and the Dirichlet boundary condition is imposed on ∂A_ϵ. We assume that $A_\epsilon = \bigcup_{j=1}^n A_\epsilon^j$, where for each j, A_ϵ^j shrink to a point $x_j \in D$, the points $x_1, \ldots, x_n$ being distinct, as $\epsilon \to 0$. This type of perturbation is also known as a "hard obstacle". In the self-adjoint case, the spectral shift has been extensively studied. In fact, a complete asymptotic expansion for the principal eigenvalue as well as for the corresponding eigenfunction are known [5, Chapter 9]. The leading order correction for all eigenvalues is also known [1]. The latter paper also includes a detailed section on the history of the problem. Very little is known on the non-selfadjoint case. In a recent paper [8], the shift for the principal eigenvalue for a large class of non-selfadjoint operators was studied. The case considered there is when the hard obstacle is a ball with respect to a local metric compatible with the operator, $V = 0$ and **BC 2**. In the present work, we elaborate the methods of [8] and develop other techniques to deal with more general setups. The core idea is that for elliptic operators which are generators of positive recurrent diffusion processes, the expected hitting time of a small set is asymptotically reciprocal to the principal eigenvalue of the operator in the punctured domain. As a byproduct, we also obtain the asymptotics for the expected hitting times of A_ϵ. The Khasminskii formula for the invariant measure of a positive recurrent diffusion links the hitting time of a small set, the invariant measure and the Newtonian capacity. In the case of **BC 2**, L_0 is indeed a generator of a positive recurrent diffusion process. In the case of **BC 1** and/or the case of a nonzero potential V, one obtains a generator of a positive recurrent process by shifting the spectrum and performing an h-transform.

The second type of perturbation considered is the "soft obstacle": instead of removing subsets, we define finite potentials on them. More precisely, we fix $x_1, \ldots, x_n \in D$ and define $W_\epsilon(x) = g(\epsilon)\sum_{j=1}^n W(|x - x_j|\epsilon^{-1})$, where W is positive with compact support and g, the scale function, is positive and nonincreasing on $(0, \infty)$. With this terminology, the hard obstacle may be considered as an infinite potential. Our discussion of the soft obstacle is restricted to the oblique-derivative Laplacian, due to the fact that an analytic expression for the shift in terms of W is more difficult to obtain when L has nonconstant coefficients. The fundamental difference between this case and the previous one is that the asymptotics are determined by the behavior of g near 0. One regime, corresponding to sufficiently fast blow-up of g near 0, coincides with the hard obstacle; hence W plays no role in this regime. It is also intuitively clear that to the other extreme, by choosing g "moderate" enough, the shift can be made arbitrarily small and will depend linearly on W. The borderline case exhibits the most interesting behavior: on the one hand the rate depends (nonlinearly) on W, but on the other hand, its dependence on ϵ is similar to that of the hard obstacle.

The third perturbation that we treat is of a periodic model of the crushed-ice problem for the Neumann-Laplacian. We recall that in the crushed-ice model, the hard obstacle consists of finitely many subsets, say identical balls, whose number goes to infinity as the diameter of each goes to zero. One approach to overcome the increasingly high complexity of this model is to assume that the centers of the balls form a sequence of identically distributed random variables; see [3] for the Dirichlet Laplacian and [6] for the Neumann-Laplacian. The asymptotic spectral shift converges to some limit in probability. For deterministic models, one may apply the method of homogenization. This is a powerful tool for studying periodic or almost period setups [4]. Here we consider a very simple model: we assume that the centers of the balls are located on lattice points of $\delta\mathbb{Z}^d$ for some $\delta > 0$ which depends on the radius of the balls. We show that the naive "limit" of the spectral shifts from the previous discussions obtained by letting the number of balls go to infinity is indeed the correct expression. This result is not new. We have decided to include it for two reasons: To provide some comparison with the subject of our study, the finite-obstacle case, and since our proof is very short. Of course, much more general results can be obtained by homogenization. Although we only consider the principal eigenvalue, the proof readily extends to the spectral shift of all the eigenvalues. Finally, we remark that this result can be compared with the estimates in [9] and [10], where a similar model was studied under a certain uniform-spacing assumption on the centers of the balls.

Let λ_c denote the principal eigenvalue for L in D with boundary condition **BC 1** or **BC 2**. Since V is bounded, λ_c is finite. Let $\widetilde{\mathrm{L}}$ denote the formal adjoint of L: $\widetilde{\mathrm{L}} = \frac{1}{2}\nabla\cdot(a\nabla) - b\cdot\nabla - \nabla\cdot b$, with a corresponding boundary condition. We denote by ϕ_c and $\widetilde{\phi_c}$ the positive eigenfunctions for L and $\widetilde{\mathrm{L}}$, respectively, corresponding to the eigenvalue λ_c, with the normalization

$$\int_D \phi_c\widetilde{\phi_c}dx = 1. \tag{1.1}$$

In what follows we write $a_\epsilon \sim b_\epsilon$ for $\lim_{\epsilon\to 0} a_\epsilon/b_\epsilon = 1$.

1.1. The hard obstacle case. Before stating the result we need some definitions. Let $x \in \mathbb{R}^d$ and let $\{H_\epsilon\}_{\epsilon>0}$ be subsets of $\mathbb{R}^d$ with smooth boundaries such that $\overline{H_\epsilon}^c$ is connected. We say that $\{H_\epsilon\}_{\epsilon>0}$ shrinks **regularly** to x if there exists a constant $K \geq 1$ such that for all $\epsilon > 0$,

$$B_{\epsilon/K}(x) \subset\subset H_\epsilon \subset\subset B_{\epsilon K}(x). \tag{1.2}$$

Let $E \subset \mathbb{R}^d$ denote a smooth domain and let $A \subset\subset E$ be an open subset such that $E\backslash\overline{A}$ is connected. We define $\mathrm{Cap}_a(A,E)$, the capacity of A in E with respect to a, by

$$\mathrm{Cap}_a(A,E) = \inf_{\{u\in C_c^\infty(E),\ u|_A\geq 1\}} \frac{1}{2}\int_E \nabla u\cdot a\nabla u dx.$$

The infimum is uniquely attained by a function known as the **capacitary potential** of A in E with respect to a. The capacitary potential is the solution to

$$\begin{cases} \frac{1}{2}\nabla\cdot(a\nabla u) = 0 & \text{in} \quad E\backslash\overline{A}; \quad \text{(a)} \\ u = 1 & \text{on} \quad \partial A; \quad \text{(b)} \\ u = 0 & \text{on} \quad \partial E. \quad \text{(c)} \end{cases} \tag{1.3}$$

Suppose that $\{H_\epsilon\}_{\epsilon>0}$ shrinks regularly to x_0, with $H_\epsilon \subset\subset B_1(x_0)$. Define

$$\mathrm{Cap}_a(H_\epsilon) = \begin{cases} \mathrm{Cap}_a(H_\epsilon, B_1(x_0)), & d = 2; \\ \lim_{R\to\infty} \mathrm{Cap}_a(H_\epsilon, B_R(x_0)), & d \geq 3. \end{cases}$$

We remark that the above expression for $\mathrm{Cap}_a(H_\epsilon)$ when $d \geq 3$ is equal to $\mathrm{Cap}_a(H_\epsilon, \mathbb{R}^d)$ and the corresponding capacitary potential is the minimal positive solution to (1.3)(a),(b) in $\mathbb{R}^d$. In the sequel, we will write $\mathrm{Cap}_d(\epsilon)$ as a short notation for $\mathrm{Cap}_{Id}(B_\epsilon(0))$. Let ω_d denote the volume of the unit ball in $\mathbb{R}^d$. It is well-known that

$$\mathrm{Cap}_d(\epsilon) = \begin{cases} \frac{\pi}{\ln \epsilon^{-1}} & d = 2; \\ \frac{d(d-2)\omega_d}{2}\epsilon^{d-2} & d \geq 3. \end{cases} \tag{1.4}$$

We are ready to state the main result:

THEOREM 1.1: *Let $\{x_1, \dots, x_n\} \subset D$ and assume that for $j = 1, \dots, n$, $\{A_\epsilon^j\}_{\epsilon>0}$ shrinks regularly to x_j. Let $A_\epsilon = \bigcup_{j=1}^n A_\epsilon^j$ and let $\lambda_{c,\epsilon}$ denote the principal eigenvalue for* L *in $D\setminus\overline{A_\epsilon}$ with the Dirichlet boundary condition on ∂A_ϵ and boundary condition* **BC 1** *or* **BC 2** *on ∂D. Then*

$$\lambda_c - \lambda_{c,\epsilon} \sim \sum_{j=1}^n \phi_c\widetilde{\phi_c}(x_j)\mathrm{Cap}_{a(x_j)}(A_\epsilon^j), \quad \text{as } \epsilon \to 0.$$

In the next theorem we find an explicit expression for the capacities appearing in Theorem 1.1 for obstacles whose shape is "compatible" with L in the sense we describe below. Given a positive definite $d \times d$ matrix Λ, let

$$\|x\|_\Lambda \equiv \frac{\sqrt{x \cdot \Lambda x}}{|\Lambda|^{\frac{1}{2d}}}, \quad x \in \mathbb{R}^d, \tag{1.5}$$

where $|\Lambda|$ denotes the determinant of Λ. This is a norm on $\mathbb{R}^d$ which preserves the standard Euclidean volume. For $\epsilon > 0$ and $x \in \mathbb{R}^d$, let $B_\epsilon^\Lambda(x)$ denote the open ball of radius ϵ centered at x with respect to the $\|\cdot\|_\Lambda$-norm. We will explicitly compute $\mathrm{Cap}_{a(x_0)}(B_\epsilon^{a^{-1}(x_0)}(x_0))$, giving us the following theorem:

THEOREM 1.2: *Let $A_\epsilon = \bigcup_{j=1}^n B_{k_j\epsilon}^{a^{-1}(x_j)}(x_j)$, where $\{x_1, \dots, x_n\} \subset D$ and $k_1, \dots, k_n$ are positive constants. Then*

$$\lambda_c - \lambda_{c,\epsilon} \sim \mathrm{Cap}_d(\epsilon) \sum_{j=1}^n \phi_c\widetilde{\phi_c}(x_j)|a(x_j)|^{1/d}k_j^{d-2}, \quad \text{as } \epsilon \to 0.$$

where $\mathrm{Cap}_d(\epsilon)$ *is as in* (1.4).

Ramark: In the case of **BC 2**, with $V = 0$ and $n = 1$, the theorem was obtained in [8].

1.2. The soft obstacle case. Here we assume that $\mathrm{L} = \frac{1}{2}\Delta$ with **BC 2**. Then ϕ_c is a positive constant and $\lambda_c = 0$. Let

$$W_\epsilon(x) = g(\epsilon)\sum_{j=1}^{n} W(|x - x_j|/\epsilon),$$

where

- $\{x_1, \ldots, x_n\} \subset D$.
- $W : [0, \infty) \to [0, \infty)$, is a nonnegative, continuous on $[0, 1]$ and vanishes outside $[0, 1]$.
- $g : (0, \infty) \to (0, \infty)$ is non-increasing and satisfies

$$[0, \infty] \ni \gamma \equiv \lim_{\epsilon \to 0} \begin{cases} g(\epsilon)\epsilon^2 \ln \epsilon^{-1} & d = 2; \\ g(\epsilon)\epsilon^2 & d \geq 3. \end{cases}$$

Recall the modified Bessel function $I_p(x) = e^{-\frac{1}{2}p\pi i} J_p(ix)$, where J_p is the Bessel function of order p. The function I_p admits the following power series representation [11, page 138]:

$$I_p(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{p+2k}}{k!\Gamma(p + k + 1)}. \tag{1.6}$$

We assume that $\phi_c \equiv 1$, in which case (1.1) implies that

$$\int_D \widetilde{\phi_c}(x)dx = 1.$$

Theorem 1.3: *Let $\lambda_{c,\epsilon}$ denote the principal eigenvalue for $\frac{1}{2}\Delta - W_\epsilon$.*

(1) *If $\gamma = 0$, then*

$$-\lambda_{c,\epsilon} \sim g(\epsilon)\epsilon^d \sum_{j=1}^{n} \widetilde{\phi_c}(x_j) \int_{B_1(0)} W(|x|)dx.$$

(2) *If $\gamma \in (0, \infty)$ and $W \equiv \beta \mathbf{1}_{[0,1]}$, $\beta > 0$, then*

$$-\lambda_{c,\epsilon} \sim \mathrm{Cap}_d(\epsilon) \sum_{j=1}^{n} \widetilde{\phi_c}(x_j) \begin{cases} (1 + \frac{1}{\beta\gamma})^{-1}, & d = 2; \\ \left(1 + \frac{d-2}{\sqrt{2\beta\gamma}} \frac{I_{d/2-1}(\sqrt{2\beta\gamma})}{I_{d/2}(\sqrt{2\beta\gamma})}\right)^{-1}, & d \geq 3. \end{cases}$$

In particular, if $d=3$*, then*

$$-\lambda_{c,\epsilon} \sim 2\pi\epsilon\left(1-\frac{\tanh(\sqrt{2\beta\gamma})}{\sqrt{2\beta\gamma}}\right)\sum_{j=1}^{n}\widetilde{\phi_c}(x_j).$$

(3) *If* $\gamma=\infty$*, then*

$$-\lambda_{c,\epsilon} \sim \mathrm{Cap}_d(\epsilon)\sum_{j=1}^{n}\widetilde{\phi_c}(x_j).$$

1.3. The crushed-ice problem. To construct the model, consider the δ-lattice consisting of the points in $\mathbb{R}^d$ of the form δq, $q\in\mathbb{Z}^d$, where δ is a positive constant. Given $r>0$, we say that a δ-lattice point in D is an r-interior point if its distance from the boundary is greater than r. For every $m\in\mathbb{N}$ we choose $\delta(m), r(m)$, such that $M(m)$, the number of $r(m)$-interior $\delta(m)$-lattice points of D, satisfies $\lim_{m\to\infty} M(m)/m=1$. Let $\{x_j^m\}_{j=1}^{M(m)}$ be an arbitrary indexing of the interior lattice points. Let K be a compact subset of $B_1(0)$, which is smooth, symmetric about the origin and such that K^c is connected. Let $A_m=\bigcup_{j=1}^{M(m)}(r(m)K+x_j^m)$. Let $\lambda_c^{(m)}$ denote the principal eigenvalue for $\frac{1}{2}\Delta$ on $D\backslash\overline{A_m}$, subject to the Neumann boundary condition on ∂D. Although we state the result only for the hard obstacle case, it remains true also for the soft obstacle case, with the appropriate changes.

Theorem 1.4: *If*

$$\beta\equiv\lim_{m\to\infty}\begin{cases} m\ln r(m)^{-1}, & d=2;\\ mr(m)^{d-2}, & d\geq 3\end{cases} \quad \textit{exists,}$$

then

$$-\lim_{m\to\infty}\lambda_c^{(m)}=\frac{\beta\mathrm{Cap}_{Id}(K)}{|D|}.$$

Ramark: Theorem 1.4 shows that the naive limit, obtained by taking the asymptotics given by Theorem 1.1 as $n\to\infty$, is achieved.

2. Proofs of Theorems 1.1 and 1.2

We begin by introducing the probabilistic machinery. Define the h-transformed operator $\mathcal{L}$ by

$$\mathcal{L}=(\mathrm{L}-\lambda_c)^{\phi_c},$$

where

$$(\mathrm{L}-\lambda_c)^{\phi_c} f = \frac{1}{\phi_c}(\mathrm{L}-\lambda_c)(\phi_c f).$$

We have

$$\mathcal{L} = \mathrm{L}_0 + a\frac{\nabla\phi_c}{\phi_c}\cdot\nabla. \tag{2.1}$$

We also denote by $\widetilde{\mathcal{L}}$ the formal adjoint of $\mathcal{L}$:

$$\widetilde{\mathcal{L}} = \frac{1}{2}\nabla\cdot(a\nabla) - \Big(b + a\frac{\nabla\phi_c}{\phi_c}Big)\cdot\nabla - \nabla\cdot\Big(b + a\frac{\nabla\phi_c}{\phi_c}\Big). \tag{2.2}$$

By the invariance of the spectrum under h-transform, it follows that the principal eigenvalue for $\mathcal{L}$ is 0. Instead of working with L directly, we will work with $\mathcal{L}$. The zeroth order term of $\mathcal{L}$ vanishes, therefore we regard $\mathcal{L}$ as the generator of a diffusion process on D. We denote by $X \equiv \{X(t) : t \geq 0\}$ the canonical process, $P_x(\cdot)$ the probability measure for the process with $X(0) = x$ for some $x \in D$, and $\mathbb{E}_x(\cdot)$ the expectation. With **BC 1**, X never hits ∂D (the drift $a\frac{\nabla\phi_c}{\phi_c}$ prevents this). With **BC 2**, X is ν-reflected at ∂D. Let $\mu = \phi_c\widetilde{\phi_c}$. Then $\widetilde{\mathcal{L}}\mu = 0$. Hence by (1.1), μ is the invariant probability density for the diffusion X. The invariant probability measure will be denoted by μ as well. As in Theorem 1.1, we denote the support of the obstacle by $A_\epsilon = \bigcup_{j=1}^n A_\epsilon^j$,where in the soft obstacle case we have $A_\epsilon^j = B_\epsilon(x_j)$. For a Borel set $E \subset \overline{D}$, we let

$$\tau_E = \inf\{t \geq 0 : X(t) \in \partial E\} \quad \text{and} \quad \tau_\epsilon = \tau_{A_\epsilon}.$$

Below, $U \subset\subset D$ is a smooth subdomain such that $\{x_1, \ldots, x_n\} \subset U$. Let $G(\cdot,\cdot)$ be the Green's function for $\mathcal{L}$ in U. Whenever one of the two variables of G is replaced with a measure, a function or a set, we should interpret the expression as the integration of that variable with respect to the corresponding measure, function or over the corresponding set (e.g., when α is a measure, $G(\alpha, y) = \int_D G(x, y)d\alpha(x)$, when A is a subset, $G(x, A) = \int_A G(x, y)dy$, etc.). For $\epsilon > 0$ sufficiently small so that $A_\epsilon \subset\subset U$, define a sequence of stopping times as follows:

$$\begin{aligned}
\sigma_1 &= \inf\{t \geq 0 : X(t) \in \partial U\},\\
\eta_k &= \inf\{t \geq \sigma_k : X(t) \in \partial A_\epsilon\}, \quad \text{and}\\
\sigma_{k+1} &= \inf\{t \geq \eta_k : X(t) \in \partial U\}.
\end{aligned}$$

The process $X(\sigma_1), X(\eta_1), \ldots$ is an irreducible Markov process on a compact state space; therefore it has a unique invariant probability measure. This implies the existence of probability measures $m_{1,\epsilon}$ on ∂A_ϵ and $m_{2,\epsilon}$ on ∂U, satisfying $P_{m_{1,\epsilon}}(X(\sigma_1) \in \cdot) = m_{2,\epsilon}$ and $P_{m_{2,\epsilon}}(X(\eta_1) \in \cdot) = m_{1,\epsilon}$. The next formula is known as the Khasminskii construction for invariant measures [2]:

$$\mu(A) = \frac{\mathbb{E}_{m_{1,\epsilon}} \int_0^{\eta_1} \mathbf{1}_A(X(s))ds}{\mathbb{E}_{m_{1,\epsilon}} \eta_1}, \qquad \text{for every Borel set } A \subset D. \tag{2.3}$$

We now derive a formula for the density of μ on A_ϵ. Let $y \in A_\epsilon$ and assume that $\delta > 0$ is sufficiently small so that $B_\delta(y) \subset A_\epsilon$. Observe that $\mathbb{E}_{m_{1,\epsilon}} \int_0^{\eta_1} \mathbf{1}_{B_\delta(y)}(X(s))ds = \mathbb{E}_{m_{1,\epsilon}} \int_0^{\sigma_1} \mathbf{1}_{B_\delta(y)}(X(s))ds$. By definition of Green's function, the right-hand side above equals $G(m_{1,\epsilon}, B_\delta(y))$. Thus, (2.3) can be written as

$$\mathbb{E}_{m_{1,\epsilon}} \eta_1 = \frac{G(m_{1,\epsilon}, B_\delta(y))}{\mu(B_\delta(y))}.$$

Letting $\delta \to 0$, we obtain

$$\mathbb{E}_{m_{1,\epsilon}} \eta_1 = \frac{G(m_{1,\epsilon}, y)}{\mu(y)}, \quad y \in A_\epsilon. \tag{2.4}$$

We state a sequence of propositions, which culminate with the proof of Theorem 1.1. Then we prove Theorem 1.2. After that we return to prove the propositions.

Proposition 2.1: *As $\epsilon \to 0$, the functions*

$$(\lambda_c - \lambda_{c,\epsilon})\mathbb{E}_x \tau_\epsilon, \quad x \in D\backslash\{x_1, \dots, x_n\}.$$

converge to 1, *uniformly on compact subsets of $D\backslash\{x_1, \dots, x_n\}$.*

We now define $\mathcal{E}_{\mathcal{L}}$, an energy functional which is an analogue in the non-selfadjoint case of the Newtonian capacity. Our strategy is to express the asymptotics for $\lambda_{c,\epsilon} - \lambda_c$ in terms of this energy functional and then by approximation of L with a constant coefficients operator show that the asymptotics of the energy can be expressed in terms of the Newtonian capacity and μ.

Let $E \subset D$ denote a smooth subdomain and let $A \subset\subset E$ be an open subset such that $E\backslash\overline{A}$ is connected. We define $\mathcal{E}_{\mathcal{L}}(A, E)$ by

$$\mathcal{E}_{\mathcal{L}}(A, E) = \frac{1}{2} \int_{E\backslash\overline{A}} (\nabla u \cdot a\nabla u)\mu dx,$$

where u is the solution to

$$\begin{cases} \mathcal{L}u = 0 & \text{in } E\backslash\overline{A}; \\ u = 1 & \text{on } \partial A; \\ u = 0 & \text{on } \partial E. \end{cases} \tag{2.5}$$

Note here that μ is uniquely determined by $\mathcal{L}$ — it is the invariant probability density for the diffusion process corresponding to $\mathcal{L}$. The function u will be called the $\mathcal{E}_{\mathcal{L}}$-capacitary potential. Let $\mathcal{L}' = \frac{1}{2}\nabla \cdot (a\nabla)$, with the co-normal boundary condition, $a\nabla u \cdot \overrightarrow{n} = 0$ on ∂D, where $\overrightarrow{n}$ is the outward unit normal to D on ∂D. Then the corresponding invariant density is $1/|D|$ and comparing the definition of $\mathcal{E}_{\mathcal{L}'}$ with the definition of Cap_a above (1.3), we observe that for $E \subset D$,

$$\mathcal{E}_{\mathcal{L}'}(A, E) = \frac{1}{|D|}\mathrm{Cap}_a(A, E).$$

Proposition 2.2:

$$\mathcal{E}_{\mathcal{L}}(A_\epsilon, U) = (\mathbb{E}_{m_{1,\epsilon}}\eta_1)^{-1} = (\mathbb{E}_{m_{2,\epsilon}}\sigma_2)^{-1}.$$

Proposition 2.3: *Let $U_j \subset\subset U$, $j = 1, \ldots, n$ be smooth domains such that $x_j \in U_j$. Let $\epsilon_0 > 0$ be such that for all $\epsilon < \epsilon_0$ and for all $j = 1, \ldots, n$, $A_\epsilon^j \subset\subset U_j$ and fix $\epsilon < \epsilon_0$.*

(1) *Let u denote the $\mathcal{E}_{\mathcal{L}}$-capacitary potential of A_ϵ in U and let $\rho_\epsilon = \max_{x \in \bigcup_{j=1}^n \partial U_j} u(x)$. Then*

$$\mathcal{E}_{\mathcal{L}}(A_\epsilon, U) \le \sum_{j=1}^n \mathcal{E}_{\mathcal{L}}(A_\epsilon^j, U_j) \le (1 - \rho_\epsilon)^{-1}\mathcal{E}_{\mathcal{L}}(A_\epsilon, U). \tag{2.6}$$

(2) *Let $\Omega \subset \mathbb{R}^d$ be a domain. If $d = 2$, assume further that Ω is bounded. Let u denote the capacitary potential of A_ϵ in Ω with respect to a and let $\rho_\epsilon = \max_{x \in \bigcup_{j=1}^n \partial U_j} u(x)$. Then*

$$\mathrm{Cap}_a(A_\epsilon, \Omega) \le \sum_{j=1}^n \mathrm{Cap}_a(A_\epsilon^j, U_j) \le (1 - \rho_\epsilon)^{-1}\mathrm{Cap}_a(A_\epsilon, \Omega) \tag{2.7}$$

In both cases (1) and (2), $\lim_{\epsilon \to 0} \rho_\epsilon = 0$.

Here is an immediate corollary:

Corollary 1: *Let $E \subset \mathbb{R}^d$ be bounded and smooth. Then*

$$\mathrm{Cap}_a(A^1_\epsilon) \sim \mathrm{Cap}_a(A^1_\epsilon, E), \quad \text{as } \epsilon \to 0.$$

Proposition 2.3 indicates that the asymptotics of $\mathcal{E}_{\mathcal{L}}(A_\epsilon, U)$ are determined by local terms. The next lemma gives a local expression for the asymptotics of $\mathcal{E}_{\mathcal{L}}(A^j_\epsilon, U_j)$. In the trivial case where a and μ are constants, such an expression can be readily obtained from the definitions. In the more general framework we employ the continuity of the coefficients of $\mathcal{L}$ and of μ to obtain good approximation for $\mathcal{E}_{\mathcal{L}}$ in terms of the Newtonian capacity for a constant coefficient operator. Comparing the definition of $\mathcal{E}_{\mathcal{L}}$ (above (2.5)) with the definition of Cap_a (above (1.3)), this type of argument should result in an asymptotic expression which is the product of $\mu(x_j)$ and $\mathrm{Cap}_{a(x_j)}(A_\epsilon)$. The next lemma makes the above heuristics precise. Although intuitively clear, the proof is tedious.

Proposition 2.4:

$$\mathcal{E}_{\mathcal{L}}(A^1_\epsilon, U) \sim \mu(x_1)\mathrm{Cap}_{a(x_1)}(A^1_\epsilon), \quad \text{as } \epsilon \to 0.$$

Theorem 1.1 follows directly from Propositions 2.1, 2.2, 2.3 and 2.4.

Proof of Theorem 1.2. Fix $x_0 \in D$. For convenience, we will assume that $x_0 = 0$ and write a for $a(0)$. In light of Theorem 1.1, it is enough to show that as $\epsilon \to 0$,

$$\mathrm{Cap}_a(B^{a^{-1}}_\epsilon(0)) \sim \begin{cases} \frac{\pi}{\ln \epsilon^{-1}}|a|^{1/2} & d = 2; \\ \frac{d(d-2)\omega_d}{2}|a|^{1/d}\epsilon^{d-2} & d \geq 3. \end{cases} \tag{2.8}$$

The proof is straightforward. By changing variables, we transform the question of the capacity of a $\|\cdot\|_{a^{-1}}$-ball into the question of the capacity of a ball with respect to the identity matrix. The answer to the latter question is well-known and is given by (1.4). Fix $\rho > 0$. Then

$$\begin{aligned} B^{a^{-1}}_\rho(0) &= \{x : x \cdot a^{-1}x < |a|^{-1/d}\rho^2\} = \{a^{1/2}y : y \cdot y < |a|^{-1/d}\rho^2\} \\ &= a^{1/2}B_{|a|^{-1/(2d)}\rho}(0). \end{aligned}$$

Therefore, if

$$Z^{a^{-1}}(\epsilon, 1) \equiv \{x : \epsilon < \|x\|_{a^{-1}} < 1\},$$

then

$$Z^{a^{-1}}(\epsilon, 1) = a^{1/2}|a|^{-\frac{1}{2d}}Z^{Id}(\epsilon, 1),$$

and we can define a bijection from the functions on $Z^{a^{-1}}(\epsilon, 1)$ to the functions on $Z^{Id}(\epsilon, 1)$ given by $v \to \widetilde{v}$, where $\widetilde{v}(y) = v(a^{1/2}|a|^{-1/(2d)}y)$. Let u be the capacitary potential for $B_\epsilon^{a^{-1}}(0)$ in $B_1^{a^{-1}}(0)$ with respect to a. Let φ be a smooth function on $\overline{Z^{a^{-1}}(\epsilon,1)}$. It follows from the change of variables formula that

$$\int_{Z^{a^{-1}}(\epsilon,1)} a\nabla u(x)\cdot\nabla\varphi(x)dx = |a|^{1/d}\int_{Z^{Id}(\epsilon,1)} \nabla\widetilde{u}(y)\cdot\nabla\widetilde{\varphi}(y)dy. \tag{2.9}$$

If $\varphi\in C_c^\infty(Z^{a^{-1}}(\epsilon,1))$, then integration by parts gives:

$$\int_{Z^{a^{-1}}(\epsilon,1)} a\nabla u(x)\cdot\nabla\varphi(x)dx = 0.$$

Thus, it follows from (2.9) that $\int_{Z^{Id}(\epsilon,1)}\nabla\widetilde{u}(y)\cdot\nabla\widetilde{\varphi}(y)dy = 0$. Since $\widetilde{\varphi}$ is arbitrary, this implies $\Delta\widetilde{u}=0$. Since $\widetilde{u}=0$ on $\partial B_1(0)$ and $\widetilde{u}=1$ on $\partial B_\epsilon(0)$, we conclude that $\widetilde{u}$ is the capacitary potential for $B_\epsilon(0)$ in $B_1(0)$ with respect to the identity matrix. By choosing $\varphi=u$ we obtain from (2.9):

$$\begin{aligned}1/2\int_{Z^{a^{-1}}(\epsilon,1)} a\nabla u(x)\cdot\nabla\varphi(x)dx &= \frac{|a|^{1/d}}{2}\int_{Z^{Id}(\epsilon,1)}|\nabla\widetilde{u}(y)|^2dy\\ &= \mathrm{Cap}_{Id}(B_\epsilon(0),B_1(0))\end{aligned}$$

Since u is the capacitary potential for $B_\epsilon^{a^{-1}}(0)$ in $B_1^{a^{-1}}(0)$ with respect to a, the left-hand side is equal to $\mathrm{Cap}_a(B_\epsilon^{a^{-1}}(0),B_1^{a^{-1}}(0))$. Thus, we have shown that

$$\mathrm{Cap}_a(B_\epsilon^{a^{-1}}(0),B_1^{a^{-1}}(0)) = |a|^{1/d}\mathrm{Cap}_{Id}(B_\epsilon(0),B_1(0)).$$

Therefore, by Corollary 1 we have

$$\mathrm{Cap}_a(B_\epsilon^{a^{-1}}(0))\sim|a|^{1/d}\mathrm{Cap}_{Id}(B_\epsilon(0)) = |a|^{1/d}\mathrm{Cap}_d(\epsilon). \tag{2.10}$$

Thus, (2.8) follows from (2.10) and (1.4). ∎

2.1. Proof of Proposition 2.1. We need a sequence of lemmas.

Lemma 2.5: *For $\epsilon>0$,*

$$\sup_{x\in D\setminus A_\epsilon}\mathbb{E}_x\tau_\epsilon<\infty.$$

Proof of Lemma 2.5. The proof depends on the particular boundary condition on ∂D.

- **BC 1**. For $\delta>0$ we define the domain D_δ,

  $$D_\delta=\{x\in\mathbb{R}^d:\mathrm{dist}(x,D)<\delta\}.$$

  We denote the principal eigenvalue for L in $D_\delta\setminus\overline{A_\epsilon}$ with the Dirichlet boundary condition on $\partial D_\delta\cup\partial A_\epsilon$ by $\lambda_{c,\epsilon,\delta}$ and let $\phi_{c,\epsilon,\delta}$ be the corresponding eigenfunction. Let $K = \max K_j$, where K_j is the constant appearing in (1.2) related to A^j_ϵ. The choice of K guarantees that $A_\epsilon \subset\subset A_{K^2\epsilon}$. By [7, Theorem 4.4.1], for every $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that

$$\lambda_{c,\epsilon,\delta} < \lambda_c.$$

It follows that

$$(\mathrm{L} - \lambda_c)\phi_{c,\epsilon,\delta} < (\lambda_{c,\epsilon,\delta} - \lambda_c)\phi_{c,\epsilon,\delta} < 0. \tag{2.11}$$

in $D \backslash \overline{A_{K^2\epsilon}}$. Denote by $\mathbb{E}^{\mathrm{L}_0}_x$ the expectation for the diffusion process on D_δ, generated by L_0 and killed at ∂D_δ, with $X(0) = x$. Let

$$v(x) = \mathbb{E}^{\mathrm{L}_0}_x \int_0^{\tau_{K^2\epsilon} \wedge \tau_D} e^{\int_0^s V(X(u))du - s\lambda_c} ds, \quad x \in D \backslash A_{K^2\epsilon}.$$

Let $G_{L-\lambda_c}$ denote the Green's function for $\mathrm{L} - \lambda_c$ on $D \backslash \overline{A_{K^2\epsilon}}$. Then, by the Feynman-Kac formula, it follows that

$$v(x) = \int_{D \backslash \overline{A_{K^2\epsilon}}} G_{L-\lambda_c}(x, y) dy. \tag{2.12}$$

From (2.11) and the Feynman-Kac formula we obtain

$$\begin{aligned} \phi_{c,\epsilon,\delta}(x) &\geq (\lambda_c - \lambda_{c,\epsilon,\delta}) \mathbb{E}^{\mathrm{L}_0}_x \int_0^{\tau_{K^2\epsilon} \wedge \tau_D} \phi_{c,\epsilon,\delta}(X(s)) e^{\int_0^s V(X(u))du - s\lambda_c} ds \\ &\geq v(x)(\lambda_c - \lambda_{c,\epsilon,\delta}) \inf_{x \in D \backslash A_{K^2\epsilon}} \phi_{c,\epsilon,\delta}(x). \end{aligned}$$

Since $\phi_{c,\epsilon,\delta}$ is continuous and positive on $\overline{D \backslash A_{K^2\epsilon}}$, it follows that v is bounded. Hence from (2.12) we find that v', defined by

$$v'(x) = \int_{D \backslash A_{K^2\epsilon}} G_{L-\lambda_c}(x, y)\phi_c(y) dy, \quad x \in D \backslash A_{K^2\epsilon},$$

is bounded. In particular, v' is the minimal positive solution to

$$\begin{cases} (\mathrm{L} - \lambda_c)v' = -\phi_c & \text{in } D \backslash A_{K^2\epsilon}; \\ v' = 0 & \text{on } \partial D \cup \partial A_{K^2\epsilon}. \end{cases}$$

From the Hopf maximum principle it follows that $\nabla \phi_c \cdot \overrightarrow{n} < 0$ and $\nabla v' \cdot \overrightarrow{n} < 0$ on ∂D, where $\overrightarrow{n}$ is the outward unit normal to D at ∂D. Let $w = v'/\phi_c$. By L'Hôpital's rule and the continuity of the first order derivatives of v' and ϕ_c up to the boundary, there exists a positive constant M such that $\limsup_{x\to\partial D} w(x) < M$. Thus,

$$\sup_{x\in D\setminus A_{K^2\epsilon}} w(x) < \infty.$$

Since w is the minimal positive solution to

$$\begin{cases} \mathcal{L}w = -1 & \text{in } D\setminus A_{K^2\epsilon} \\ w = 0 & \text{on } \partial A_{K^2\epsilon}, \end{cases}$$

we have that $w(x) = \mathbb{E}_x \tau_{K^2\epsilon}$ and the lemma follows.

- **BC 2**. ϕ_c is strictly positive in $\overline{D}$. This implies that $\mathcal{L}$ is uniformly elliptic and has a bounded drift coefficient. Therefore,

$$\rho = \sup_{x\in D\setminus A_\epsilon} P_x(\tau_\epsilon > 1) < 1.$$

By the Markov property

$$\mathbb{E}_x\tau_\epsilon < \sum_{k=0}^{\infty} P_x(\tau_\epsilon > k) < \sum_{k=0}^{\infty} \rho^k < \infty.$$

The bound on the right-hand side is independent of the specific choice of x. This implies the lemma. ∎

Lemma 2.6: *Let $f \in C^\alpha(\overline{D})$, be a function with $0 \le f \le 1$ which is not identically 0. Let*

$$u_\epsilon(x) = \frac{\mathbb{E}_x \int_0^{\tau_\epsilon} f(X(s))ds}{\mathbb{E}_{m_{2,\epsilon}} \int_0^{\tau_\epsilon} f(X(s))ds}, \quad x \in D\setminus\{x_1,\dots,x_n\}.$$

Then $\lim_{\epsilon\to 0} u_\epsilon(x) = 1$, uniformly on compacts.

Corollary 2: *The function*

$$\frac{\mathbb{E}_x\tau_\epsilon}{\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon}, \quad x \in D\setminus\{x_1,\dots,x_n\}$$

converges to 1, uniformly on compacts.

Proof of Lemma 2.6. Let

$$v_\epsilon(x) = \mathbb{E}_x \int_0^{\tau_\epsilon} f(X(s))ds, \quad x \in D\setminus\overline{A_\epsilon}.$$

Then

$$u_\epsilon = \frac{v_\epsilon}{\int v_\epsilon dm_{2,\epsilon}}.$$

We will prove below that there exists a positive constant K, independent of ϵ, such that

$$v_\epsilon \leq K\left(1 + \int v_\epsilon dm_{2,\epsilon}\right). \tag{2.13}$$

Since X is recurrent, $\lim_{\epsilon\to 0}\int v_\epsilon dm_{2,\epsilon} = \infty$ and it follows from (2.13) that $\{u_\epsilon\}$ is uniformly bounded. Since $\mathcal{L}v_\epsilon = -f$, it follows that $\{\mathcal{L}u_\epsilon\}$ is bounded in $C^\alpha(\overline{D})$. By the standard compactness argument involving the Schauder estimates and the Ascoli-Arzella theorem, it follows that $\{u_\epsilon\}$ is precompact in $C_2^{loc}(D\backslash\{x_1,\dots,x_n\})$. Let u be a limit obtained by some subsequence. It is clear that $\mathcal{L}u = 0$ in $D\backslash\{x_1,\dots,x_n\}$ and that u is bounded. We note, however, that this argument does not imply automatically that u satisfies the boundary condition (in **BC 2**). Fix $x \neq y \in D\backslash\overline{A_\epsilon}$ and let $\epsilon', \rho > 0$. Then

$$u_\epsilon(x) = \mathbb{E}_x u_\epsilon(X(\tau_{B_\rho(y)} \wedge \tau_{\epsilon'})) + \frac{\mathbb{E}_x \int_0^{\tau_{B_\rho(y)}\wedge\tau_{\epsilon'}} f(X(s))ds}{\int v_\epsilon dm_{2,\epsilon}},$$

for all ϵ sufficiently small. Letting $\epsilon \to 0$, it follows that

$$u(x) = \mathbb{E}_x u(X(\tau_{B_\rho(y)} \wedge \tau_{\epsilon'})).$$

Letting $\epsilon' \to 0$, we obtain $u(x) = \mathbb{E}_x u(X(\tau_{B_\rho(y)}))$. Finally, letting $\rho \to 0$, we have $u(x) = u(y)$. Hence u is a constant function. We will now show that $u = 1$. Indeed, by the triangle inequality,

$$\int u_\epsilon dm_{2,\epsilon} - \int |u - u_\epsilon| d_{m_{2,\epsilon}} \leq \int u dm_{2,\epsilon} \leq \int u_\epsilon dm_{2,\epsilon} + \int |u - u_\epsilon| dm_{2,\epsilon}.$$

As $\int u_\epsilon dm_{2,\epsilon} = 1$, letting $\epsilon \to 0$ gives the required result. Since $\{u_\epsilon\}$ is precompact in C^2_{loc} and every convergent subsequence converges to 1, it follows that $\{u_\epsilon\}$ converges to 1 in C^2_{loc}, as $\epsilon \to 0$.

We now return to the proof of (2.13). Fix a domain E satisfying $\{x_1,\dots,x_n\} \subset E \subset\subset U$. By the strong Markov property,

$$v_\epsilon(x) = \mathbb{E}_x \int_0^{\tau_E} f(X(s))ds + \mathbb{E}_x \mathbb{E}_{X(\tau_E)} \int_0^{\tau_\epsilon} f(X(s))ds, \quad x \in D\backslash E.$$

Since f takes values in $[0,1]$, the first term on the left-hand side is bounded above by $C_1 \equiv \max_{x\in\partial U} \mathbb{E}_x \tau_E < \infty$, due to Lemma 2.5 The second term on the right-hand side is an $\mathcal{L}$-harmonic function on $U\backslash\overline{E}$, since it is equal to $\mathbb{E}_x h(X(\tau_E))$ with $h(y) \equiv \mathbb{E}_y \int_0^{\tau_\epsilon} f(X(s))ds$. By Harnack's inequality,

$$\mathbb{E}_x \mathbb{E}_{X(\tau_E)} \int_0^{\tau_\epsilon} f(X(s))ds \leq C_2 \mathbb{E}_y \mathbb{E}_{X(\tau_E)} \int_0^{\tau_\epsilon} f(X(s))ds, \quad x, y \in \partial U,$$

where $C_2 \geq 1$ is a constant independent of ϵ. By integrating the right-hand side with respect to $m_{2,\epsilon}$ we obtain

$$\begin{aligned}\max_{x\in\partial U}\mathbb{E}_x\mathbb{E}_{X(\tau_E)}\int_0^{\tau_\epsilon} f(X(s))ds &\leq C_2\mathbb{E}_{m_{2,\epsilon}}\mathbb{E}_{X(\tau_E)}\int_0^{\tau_\epsilon} f(X(s))ds\\ &\leq C_2\mathbb{E}_{m_{2,\epsilon}}\int_0^{\tau_\epsilon} f(X(s))ds\\ &= C_2\int v_\epsilon dm_{2,\epsilon}.\end{aligned}$$

Summarizing the last observations, we have proved that

$$\max_{x\in\partial U} v_\epsilon(x) \leq C_1 + \mathbb{E}_{m_{2,\epsilon}}\int_0^{\tau_\epsilon} f(X(s))ds = C_1 + C_2\int v_\epsilon dm_{2,\epsilon}. \tag{2.14}$$

We now derive (2.13) from (2.14). Since $\mathcal{L}v_\epsilon = -f$, it follows that

$$v_\epsilon(x) = \mathbb{E}_x v_\epsilon(X(\tau_U\wedge\tau_\epsilon)) + \mathbb{E}_x\int_0^{\tau_U\wedge\tau_\epsilon} f(X(s))ds. \tag{2.15}$$

Let $C_3 = \sup_{x\in D}\mathbb{E}_x\tau_U$. Then $C_3<\infty$, due to Lemma 2.5. It is clear that the second term on the right-hand side of (2.15) is bounded above by C_3. Since v_ϵ vanishes on ∂A_ϵ, the first term on the right-hand side of (2.15) is bounded above by $\max_{x\in\partial U} v_\epsilon(x)$. Therefore, it follows from (2.14) that:

$$v_\epsilon(x)\leq C_1 + C_2\int v_\epsilon dm_{2,\epsilon} + C_3,$$

completing the proof of (2.13). ∎

The next lemma will be also employed in the proof for the soft obstacle case, Theorem 1.3. In order to unify notation, we need an additional definition. Let $\mathcal{L}_\epsilon$ denote the perturbation of $\mathcal{L}$. For the hard obstacle, $\mathcal{L}_\epsilon$ corresponds to $\mathcal{L}$ on the domain $D\backslash\overline{A_\epsilon}$, with the Dirichlet boundary condition on ∂A_ϵ. In the soft obstacle case (see Section 1.2), $\mathcal{L}_\epsilon = (\mathrm{L}-\lambda_c - W_\epsilon)^{\phi_c} = \mathcal{L} - W_\epsilon$ on D. The principal eigenvalue for $\mathcal{L}_\epsilon$ is $\lambda_{c,\epsilon}-\lambda_c$.

Lemma 2.7: *Let ϕ_ϵ denote the positive eigenfunction corresponding to $\lambda_{c,\epsilon}-\lambda_c$, the principal eigenvalue for $\mathcal{L}_\epsilon$, normalized by*

$$\int_{\partial U}\phi_\epsilon(x)dm_{2,\epsilon}(x) = 1. \tag{2.16}$$

Then $\{\phi_\epsilon\}_{\epsilon>0}$ is uniformly bounded and converges to 1 in $C^2_{loc}(D\backslash\{x_1,\dots,x_n\})$, as $\epsilon\to 0$.

Proof of Lemma 2.7. Recall that in contrast to the other lemmas, which are to be proved only for the hard obstacle case, this one we need to prove both for the hard and the soft obstacle. We write λ_ϵ as a short notation for $\lambda_{c,\epsilon} - \lambda_c$. Note that $\lambda_\epsilon < 0$. In the hard obstacle case we extend ϕ_ϵ to D continuously, by letting $\phi_\epsilon \equiv 0$ on A_ϵ. The first step in the proof is to show that $\{\phi_\epsilon\}_{\epsilon>0}$ is uniformly bounded. Let $E \subset\subset D$ be a smooth subdomain, such that $U \subset E$. By Lemma 2.5 and the Chebychev inequality, there exists some $k \in \mathbb{N}$ such that

$$\rho \equiv \ln \sup_{x \in D\setminus E} P_x(\tau_E > k) < 0.$$

Since $\lim_{\epsilon\to 0} \lambda_\epsilon = 0$, we have $\rho - k\lambda_\epsilon < 0$, for $\epsilon > 0$ sufficiently small. Using the identity $\mathbb{E}_x e^{-\lambda_\epsilon \tau_E} = 1 - \lambda_\epsilon \int_0^\infty P_x(\tau_U > s) e^{-\lambda_\epsilon s} ds$, we obtain

$$\mathbb{E}_x e^{-\lambda_\epsilon \tau_E} \le 1 - \lambda_\epsilon k \sum_{j=0}^{\infty} P_x(\tau_E > kj) e^{-\lambda_\epsilon (j+1)k}.$$

By the Markov property it then follows that

$$M \equiv \limsup_{\epsilon\to 0} \sup_{x\in D\setminus E} \mathbb{E}_x e^{-\lambda_\epsilon \tau_E} \le \limsup_{\epsilon\to 0} (1 - \lambda_\epsilon k \sum_{j=0}^{\infty} e^{j\rho - (j+1)k\lambda_\epsilon}) < \infty. \tag{2.17}$$

Therefore, $\mathbb{E}_x \phi_\epsilon(X(\tau_E)) e^{-\lambda_\epsilon \tau_E}$ defines a positive solution to the equation $\mathcal{L}u = \lambda_\epsilon u$ in $D\setminus\overline{E}$. By the Feynman–Kac formula, ϕ_ϵ admits the representation

$$\phi_\epsilon(x) = \mathbb{E}_x \phi_\epsilon(X(\tau_E)) e^{-\lambda_\epsilon \tau_E}, \quad x \in D\setminus E. \tag{2.18}$$

By (2.16) and the Harnack inequality, there exists a constant $c \ge 1$, such that

$$\limsup_{\epsilon\to 0} \sup_{y\in\partial E} \phi_\epsilon(y) \le c. \tag{2.19}$$

From (2.17), (2.18) and (2.19) we get

$$\limsup_{\epsilon\to 0} \sup_{x\in D\setminus E} \phi_\epsilon(x) \le cM. \tag{2.20}$$

Let $E_1 \subset\subset D$ be a smooth subdomain such that $E \subset\subset E_1$ and let ρ and φ denote the principal eigenvalue and eigenfunction for $\mathcal{L}$ in E_1 with the Dirichlet boundary condition on ∂E_1. Since $\rho < 0$, there is no loss of generality assuming that $\lambda_\epsilon > \rho$. On E_1, denote the h-transformed operator

$$(\mathcal{L}_\epsilon - \lambda_\epsilon)^\varphi = \mathcal{L}_\epsilon + a\frac{\nabla\varphi}{\varphi} \cdot \nabla - \lambda_\epsilon + \rho.$$

In the hard obstacle case, the zeroth order term of $(\mathcal{L}_\epsilon - \lambda_\epsilon)^\varphi$ is $\rho - \lambda_\epsilon$, which is strictly negative. In the soft obstacle case, this term is $-W_\epsilon + \rho - \lambda_\epsilon$ which is also strictly negative. Furthermore, ϕ_ϵ/φ is $(\mathcal{L}_\epsilon - \lambda_\epsilon)^\varphi$-harmonic on the respective domains, which are $E\backslash\overline{A_\epsilon}$ for the hard obstacle and E for the soft obstacle. The maximum principle gives

$$\max_{x\in\overline{E}} \frac{\phi_\epsilon}{\varphi}(x) = \max_{x\in\partial E} \frac{\phi_\epsilon}{\varphi}(x) \tag{2.21}$$

Let $M' = \frac{\sup_{x\in E}\varphi(x)}{\inf_{x\in E}\varphi(x)} < \infty$. Then,

$$\limsup_{\epsilon\to 0} \max_{x\in\overline{E}} \phi_\epsilon(x) \le M' \max_{x\in\partial E} \phi_\epsilon(x) \le M'c, \tag{2.22}$$

where the first inequality follows from (2.21) and the second one follows from (2.19). Now (2.22) and (2.20) imply that indeed $\{\phi_\epsilon\}_{\epsilon>0}$ is uniformly bounded.

Due to the uniform boundedness, the standard compactness argument implies that $\{\phi_\epsilon\}_{\epsilon>0}$ is precompact in the C^2_{loc}-norm. In particular, the convergence is uniform on compacts. Let ϕ be a limit obtained by some subsequence. Clearly $\mathcal{L}\phi = 0$ in $D\backslash\{x_1,\ldots,x_n\}$. The positivity of ϕ is guaranteed from (2.16), the Harnack inequality and the uniform convergence on compacts. Let $x \neq y$ be in $D\backslash\{x_1,\ldots,x_n\}$. Since ϕ is bounded and since $\{x_1,\ldots,x_n\}$ are polar for the $\mathcal{L}$-diffusion, we see that $\phi(x) = \mathbb{E}_x\phi(X(\tau_{B_\rho(y)}))$, for all $\rho > 0$ sufficiently small. By the bounded convergence theorem, $\phi(x) = \lim_{\rho\to 0}\mathbb{E}_x\phi(X(\tau_{B_\rho(y)})) = \phi(y)$. Then by (2.16), $\phi = 1$. Hence 1 is the limit of every convergent subsequence. This implies that $\{\phi_\epsilon\}_{\epsilon>0}$ converges to 1 in C^2_{loc}. ∎

We return to the hard obstacle case and prove Proposition 2.1. We adopt the notation from Lemma 2.7. Since $\mathcal{L}\phi_\epsilon = (\lambda_{c,\epsilon} - \lambda_c)\phi_\epsilon$ on $D\backslash\overline{A_\epsilon}$ and $\phi_\epsilon = 0$ on ∂A_ϵ, we observe that

$$\phi_\epsilon(x) = (\lambda_c - \lambda_{c,\epsilon})\mathbb{E}_x \int_0^{\tau_\epsilon} \phi_\epsilon(X(t))dt.$$

Integrating both sides above with respect to $dm_{2,\epsilon}$, it follows from (2.16) that

$$(\lambda_c - \lambda_{c,\epsilon})^{-1} = \mathbb{E}_{m_{2,\epsilon}} \int_0^{\tau_\epsilon} \phi_\epsilon(X(t))dt.$$

By Khasminskii's construction in (2.3) and the fact that for all $f \in C(\overline{D})$, $\mathbb{E}_{m_{2,\epsilon}} \int_0^{\sigma_2} f(X(t))dt = \mathbb{E}_{m_{1,\epsilon}} \int_0^{\eta_1} f(X(t))dt$, we obtain

$$\begin{aligned}(\lambda_c - \lambda_{c,\epsilon})^{-1} &= \mathbb{E}_{m_{2,\epsilon}} \int_0^{\sigma_2} \phi_\epsilon(X(s))ds - \mathbb{E}_{m_{1,\epsilon}} \int_0^{\sigma_1} \phi_\epsilon(X(s))ds \\ &= \Big(\int \phi_\epsilon d\mu\Big)\mathbb{E}_{m_{2,\epsilon}}\sigma_2 - \mathbb{E}_{m_{1,\epsilon}} \int_0^{\sigma_1} \phi_\epsilon(X(s))ds.\end{aligned}$$

As $\epsilon \to 0$, ϕ_ϵ are uniformly bounded due to Lemma 2.7, and $\mathbb{E}_{m_{1,\epsilon}}\sigma_1$ are clearly also uniformly bounded. Thus,

$$(\lambda_c - \lambda_{c,\epsilon})^{-1} = \Big(\int \phi_\epsilon d\mu\Big)\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon + O(1), \text{ as } \epsilon \to 0.$$

We conclude that

$$(\lambda_c - \lambda_{c,\epsilon})\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon = \Big(\int \phi_\epsilon d\mu + o(1)\Big)^{-1}.$$

By Lemma 2.7 and the bounded convergence theorem, the right-hand side goes to 1 as $\epsilon \to 0$. Thus, Proposition 2.1 follows directly from Corollary 2.

Throughout Sections 2.2 and 2.3 below, we denote the unit outward normal to $U\backslash\overline{A_\epsilon}$ on $\partial A_\epsilon \cup \partial U$ by $\overrightarrow{n}$.

2.2. Proof of Proposition 2.2. We prove the proposition via a sequence of lemmas. We begin with some notation. We denote the Green's function for $\mu\mathcal{L}$ by $g(\cdot,\cdot)$. Clearly, $g(x,y) = \frac{G(x,y)}{\mu(y)}$. As before, whenever one of the two variables of g is replaced with a measure, a function or a set, we should interpret the expression as the integration of that variable with respect to the corresponding measure, function or over the corresponding set (e.g., when α is a measure, $g(\alpha, y) = \int_D g(x,y)d\alpha(x)$, when A is a subset, $g(x,A) = \int_A g(x,y)dy$, etc.). We also define $\widetilde{\mu\mathcal{L}}$, the formal adjoint of $\mu\mathcal{L}$. Then

$$\widetilde{\mu\mathcal{L}} = \frac{1}{2}\nabla\cdot(a\nabla) - \mu\Big(b + a\frac{\nabla\phi_c}{\phi_c}\Big)\cdot\nabla - \nabla\cdot\Big(\mu\Big(b + a\frac{\nabla\phi_c}{\phi_c}\Big)\Big).$$

Let

$$\overline{b} = -\frac{1}{2}a\nabla\mu + \mu\Big(b + a\frac{\nabla\phi}{\phi}\Big).$$

The operator $\widetilde{\mathcal{L}}$ was defined in (2.2). It satisfies: $\nabla\cdot\overline{b} = -\widetilde{\mathcal{L}}\mu = 0$. Consequently, we have

$$\mu\mathcal{L} = \nabla\cdot(\frac{1}{2}(\mu a\nabla) + \overline{b}) = \frac{1}{2}\nabla\cdot(\mu a\nabla) + \overline{b}\cdot\nabla, \text{ and}$$
$$\widetilde{\mu\mathcal{L}} = \nabla\cdot(\frac{1}{2}(\mu a\nabla) - \overline{b}) = \frac{1}{2}\nabla\cdot(\mu a\nabla) - \overline{b}\cdot\nabla.$$

Lemma 2.8: *Let v_ϵ be the solution to*

$$\begin{cases} \widetilde{\mu\mathcal{L}}v = 0 & \text{on } U\backslash\overline{A_\epsilon}; \\ v = \mathbb{E}_{m_{1,\epsilon}}\eta_1 & \text{on } \partial A_\epsilon; \\ v = 0 & \text{on } \partial U. \end{cases}$$

Then

(1) $dm_{1,\epsilon}(x) = \frac{1}{2}\mu a \nabla v_\epsilon(x) \cdot \overrightarrow{n}\, dx$, *and*
(2) $v_\epsilon(y) = g(m_{1,\epsilon}, y)$ *for all* $y \in U \backslash \overline{A_\epsilon}$.

Proof. We define

$$\rho_\epsilon(x) = \frac{1}{2}\mu a \nabla v_\epsilon(x) \cdot \overrightarrow{n}(x), \quad x \in \partial A_\epsilon. \tag{2.23}$$

Thus, ρ_ϵ can be interpreted as the density of a finite measure supported on ∂A_ϵ with respect to the surface area. We will denote this measure by ρ_ϵ as well. Fix $y \in U \backslash \overline{A_\epsilon}$ and for every $\delta > 0$ sufficiently small, we let φ_δ be a smooth, nonnegative function, supported on $B_{2\delta}(y)$, which is identically 1 on $B_\delta(y)$. Clearly, $\mu \mathcal{L} g(x, \varphi_\delta) = -\varphi_\delta(x)$. Hence,

$$\int_{U \backslash \overline{A_\epsilon}} \varphi_\delta(x) v_\epsilon(x) dx = -\int_{U \backslash \overline{A_\epsilon}} \mu \mathcal{L} g(x, \varphi_\delta) v_\epsilon(x) dx.$$

In particular, it follows that

$$v_\epsilon(y) = -\lim_{\delta \to 0} \int_{U \backslash \overline{A_\epsilon}} \mu \mathcal{L} g(x, \varphi_\delta) v_\epsilon(x) dx.$$

We integrate by parts the right-hand term:

$$\begin{aligned}\int_{U \backslash \overline{A_\epsilon}} \mu \mathcal{L} g(x, \varphi_\delta) v_\epsilon(x) dx = & \int_{U \backslash \overline{A_\epsilon}} g(x, \varphi_\delta) \widetilde{\mu \mathcal{L}} v_\epsilon(x) dx \\ & + \int_{\partial A_\epsilon} \left(\frac{1}{2}\mu a \nabla g(x, \varphi_\delta) + \overline{b} g(x, \varphi_\delta) \right) v_\epsilon(x) \cdot \overrightarrow{n}\, dx \\ & - \int_{\partial A_\epsilon} \frac{1}{2}\mu a \nabla v_\epsilon(x) g(x, \varphi_\delta) \cdot \overrightarrow{n}\, dx.\end{aligned}$$

The first term on the right-hand side is equal to 0, because $\widetilde{\mu \mathcal{L}} v_\epsilon = 0$. Since v_ϵ is constant on ∂A_ϵ and $\mu \mathcal{L} = \nabla \cdot (\frac{1}{2}\mu a \nabla + \overline{b})$, the divergence theorem for the domain A_ϵ implies that the second term is equal to $\int_{A_\epsilon} \varphi_\delta(x) dx$. Our choice of φ_δ guarantees that as δ tends to 0, this quantity converges to 0. The last term is equal to

$$-\int_U \varphi_\delta(z) \int_{\partial A_\epsilon} \frac{1}{2}\mu a \nabla v_\epsilon(x) g(x, z) \cdot \overrightarrow{n}\, dx dz = -\int_U \varphi_\delta(z) g(\rho_\epsilon, z) dz,$$

whose limit as δ tends to 0 is $-g(\rho_\epsilon, y)$. Therefore we have proved that $v_\epsilon(y) = g(\rho_\epsilon, y)$ for all $y \in U \backslash \overline{A_\epsilon}$. In order to complete the proof, we have to show that $\rho_\epsilon = m_{1,\epsilon}$.

The function $g(\rho_\epsilon, \cdot)$ is $\widetilde{\mu\mathcal{L}}$-harmonic on A_ϵ and is identically $\mathbb{E}_{m_{1,\epsilon}}\eta_1$ on ∂A_ϵ. Since $g(x,y) = \frac{G(x,y)}{\mu(y)}$, (2.4) implies that $g(\rho_\epsilon, y) = g(m_{1,\epsilon}, y)$ on ∂A_ϵ. Since both functions vanish on ∂U, it follows from the maximum principle that they are equal on $U\backslash A_\epsilon$. By the maximum principle for each of the domains A_ϵ^j, $j = 1, \ldots, n$, it follows that $g(\rho_\epsilon, y) = g(m_{1,\epsilon}, y) = E_{m_{1,\epsilon}}\eta_1$ on A_ϵ. Let $\gamma = \rho_\epsilon - m_{1,\epsilon}$. Then γ is a signed measure, supported on ∂A_ϵ and $g(\gamma, y)$ is identically 0 on A_ϵ. Let $A \subset\subset U$ be a compact subset of U and let $\{f_k\} \subset C_c^2(U)$ be a bounded sequence which converges to $\mathbf{1}_A$ pointwise. Since

$$\int g(x,y)\mu\mathcal{L}f_k(y)dy = -f_k(x),$$

integration of both sides with respect to γ yields:

$$\int g(\gamma,y)\mu\mathcal{L}f_k(y)dy = -\int f_k(x)d\gamma(x).$$

By assumption, the left-hand side equals 0. The right-hand side converges to $-\gamma(A)$, as $k \to \infty$. This implies that $\gamma \equiv 0$, concluding the proof. ∎

Next we show that $\mathcal{E}_\mathcal{L}$ has a representation as a surface integral on ∂A_ϵ:

Lemma 2.9: *Let u be the $\mathcal{E}_\mathcal{L}$-capacitary potential of A_ϵ in U. Then $\mathcal{E}_\mathcal{L}(A_\epsilon, U) = \frac{1}{2}\int_{\partial A_\epsilon} \mu a \nabla u \cdot \vec{n}\, dx$.*

Proof. The proof is straightforward. Since $u = 1$ on ∂A_ϵ and $\nabla \cdot \overline{b} = 0$, by the divergence theorem $\int_{\partial A_\epsilon} \overline{b}u^2 \cdot \vec{n}\, dx = 0$. Since $u = 0$ on ∂U, we obtain

$$0 = \int_{\partial A_\epsilon} \overline{b}u^2 \cdot \vec{n}\, dx = \int_{\partial A_\epsilon \cup \partial U} \overline{b}u^2 \cdot \vec{n}\, dx = \int_{U\backslash \overline{A_\epsilon}} \nabla \cdot (\overline{b}u^2) dx. \tag{2.24}$$

Consequently,

$$\frac{1}{2}\int_{\partial A_\epsilon} \mu a \nabla u \cdot \vec{n}\, dx = \int_{\partial A_\epsilon \cup \partial U} \left(\frac{1}{2}\mu a \nabla u + \overline{b}u\right) u \cdot \vec{n}\, dx.$$

Thus,

$$\frac{1}{2}\int_{\partial A_\epsilon} \mu a \nabla u \cdot \vec{n}\, dx = \int_{U\backslash \overline{A_\epsilon}} \nabla \cdot (\frac{1}{2}\mu a \nabla u + \overline{b}u) u dx + \int_{U\backslash \overline{A_\epsilon}} \frac{1}{2}\mu a \nabla u \cdot \nabla u dx$$
$$+ \int_{U\backslash \overline{A_\epsilon}} \frac{1}{2}\nabla \cdot (\overline{b}u^2) dx,$$

where for the third term on the right-hand side we have used the fact that $\nabla \cdot (\overline{b}u^2) = \overline{b} \cdot \nabla u^2$, because $\nabla \cdot \overline{b} = 0$. The first term on the right-hand side is

$\int_{U\setminus\overline{A_\epsilon}}(\mu\mathcal{L}u)udx = 0$. The second term is $\mathcal{E}_{\mathcal{L}}(A_\epsilon, U)$. The third term is equal to 0 due to (2.24). ∎

We are ready to prove Proposition 2.2. Let v_ϵ be as in Lemma 2.8 and let u be as in Lemma 2.9. Noting that the restriction of u and v_ϵ to $\partial A_\epsilon \cup \partial U$ are equal, we may apply (2.24) here to obtain $\int_{\partial A_\epsilon\cup\partial U}\overline{b}v_\epsilon u\cdot\overrightarrow{n}\,dx = 0$. Then, by lemma 2.8,

$$1 = m_{1,\epsilon}(\partial A_\epsilon) = \int_{\partial A_\epsilon}\frac{1}{2}\mu a\nabla v_\epsilon u\cdot\overrightarrow{n}\,dx = \int_{\partial A_\epsilon\cup\partial U}\left(\frac{1}{2}\mu a\nabla v_\epsilon - \overline{b}v_\epsilon\right)u\cdot\overrightarrow{n}\,dx.$$

Consequently,

$$1 = \int_{U\setminus\overline{A_\epsilon}}\nabla\cdot\left(\frac{1}{2}\mu a\nabla v_\epsilon - \overline{b}v_\epsilon\right)udx + \int_{U\setminus\overline{A_\epsilon}}\frac{1}{2}\mu a\nabla v_\epsilon\cdot\nabla udx - \int_{U\setminus\overline{A_\epsilon}}v_\epsilon\overline{b}\cdot\nabla udx.$$

The first term on the right-hand side is $\int_{U\setminus\overline{A_\epsilon}}u\widetilde{\mu\mathcal{L}}v_\epsilon dx = 0$. Since $\frac{1}{2}\mu a\nabla v_\epsilon\cdot\nabla u = \frac{1}{2}\nabla\cdot(v_\epsilon\mu a\nabla u) - \frac{1}{2}\nabla\cdot(\mu a\nabla u)v_\epsilon$, it follows that the last two terms are equal to

$$\int_{\partial A_\epsilon}v_\epsilon\frac{1}{2}\mu a\nabla u\cdot\overrightarrow{n}\,dx - \int_{U\setminus\overline{A_\epsilon}}\nabla\cdot\left(\frac{1}{2}(\mu a\nabla u) + \overline{b}u\right)v_\epsilon dx.$$

By the definitions of v_ϵ and $\mathcal{E}_{\mathcal{L}}(A_\epsilon,U)$, the first term is equal to $\mathbb{E}_{m_{1,\epsilon}}\eta_1\mathcal{E}_{\mathcal{L}}(A_\epsilon,U)$. As for the second term, it is equal to $-\int_{U\setminus\overline{A_\epsilon}}v_\epsilon\mu\mathcal{L}udx = 0$. Thus, we have shown that $1 = \mathbb{E}_{m_{1,\epsilon}}\eta_1\mathcal{E}_{\mathcal{L}}(A_\epsilon, U)$. The proof is now complete because $\mathbb{E}_{m_{1,\epsilon}}\eta_1 = \mathbb{E}_{m_{2,\epsilon}}\sigma_2$.

2.3. Proof of Propositions 2.3 and 2.4.

Proof of Proposition 2.3. We prove both cases simultaneously. For (2.6), we let u_j, $j = 1,\dots,n$ denote the $\mathcal{E}_{\mathcal{L}}$-capacitary potential of A_ϵ^j in U_j. For (2.7), u_j is the capacitary potential of A_ϵ^j in U_j with respect to a. We remark that the assumption that Ω is bounded if $d = 2$ was made to insure that u is not constant.

Since $u = u_j = 1$ on ∂A_ϵ^j, ∇u_j and ∇u are positive, scalar multiples of $\overrightarrow{n}$, the outward unit normal to $U\setminus\overline{A_\epsilon^j}$ on ∂A_ϵ^j. By the maximum principle, $0 < u_j \le u < 1$ on $U_j\setminus\overline{A_\epsilon}^j$ hence $\nabla u_j\cdot\overrightarrow{n} \ge \nabla u\cdot\overrightarrow{n}$. Since a is positive definite,

$$a\nabla u_j\cdot\overrightarrow{n} \ge a\nabla u\cdot\overrightarrow{n}. \tag{2.25}$$

Let $v_j = (1-\rho_\epsilon)u_j + \rho_\epsilon$. Now $u = v_j = 1$ on ∂A_ϵ^j and $u \le v_j$ on ∂U_j. Again, by the maximum principle, $0 < u \le v_j < 1$ on $U_j\setminus\overline{A_\epsilon^j}$. Hence

$$(1-\rho_\epsilon)a\nabla u_j\cdot\overrightarrow{n} = a\nabla v_j\cdot\overrightarrow{n} \le a\nabla u\cdot\overrightarrow{n}. \tag{2.26}$$

In light of Lemma 2.9, (2.6) follows from (2.25) and (2.26). It is easy to see that $\mathrm{Cap}_a(A_\epsilon,\Omega)$ admits a similar representation to the one given by Lemma 2.9:

$$\mathrm{Cap}_a(A_\epsilon,\Omega)=\frac{1}{2}\int_{\partial A_\epsilon} a\nabla u\cdot\overrightarrow{n}\,dx.$$

Hence, (2.7) follows from (2.25) and (2.26) as well. Finally, by the standard compactness argument, $u\to 0$ uniformly on compact subsets of $U\backslash\{x_1,\dots,x_n\}$, as $\epsilon\to 0$. Hence, $\lim_{\epsilon\to 0}\rho_\epsilon=0$. ∎

Proof of Proposition 2.4. Without loss of generality we may assume that $x_1=0$. To simplify notation, we write A_ϵ instead of A^1_ϵ. We will prove that there exists a constant $C>0$ such that for every $\delta>0$ sufficiently small there exists a domain $U_\delta\subset\subset D$ containing 0 with the property

$$1-\delta C\le\frac{\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon,U_\delta)}{\mathcal{E}(A_\epsilon,U_\delta)}\le 1+\delta C,\qquad\text{for all sufficiently small }\epsilon. \tag{2.27}$$

Once (2.27) is established, we can apply Corollary 1 to obtain

$$1-\delta C\le\liminf_{\epsilon\to0}\frac{\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon)}{\mathcal{E}_{\mathcal{L}}(A_\epsilon,U_\delta)}\le\limsup_{\epsilon\to0}\frac{\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon)}{\mathcal{E}_{\mathcal{L}}(A_\epsilon,U_\delta)}\le 1+\delta C.$$

Since δ is arbitrary, this concludes the proof of the proposition.

We now prove (2.27). Let $\hat b=b+\frac{a\nabla\phi_c}{\phi_c}$. Then $\mathcal{L}=\frac12\nabla\cdot(a\nabla)+\hat b\cdot\nabla$. We define the constant-coefficients operator

$$\mathcal{L}_0=\frac{1}{2}\nabla\cdot(a(0)\nabla)+\hat b(0)\cdot\nabla.$$

Since $a(0)$ is an invertible matrix, by letting $Q(x)=a(0)^{-1}\hat b(0)\cdot x$, we have $\hat b(0)=a(0)\nabla Q$ and

$$\mathcal{L}_0=\frac{1}{2}e^{-2Q}\nabla\cdot(e^{2Q}a(0)\nabla).$$

Fix a subdomain $U_0\subset\subset D$ such that $0\in U_0$. We choose a constant $C_1>0$ such that the following conditions will hold:

$$|v|^2\le C_1 v\cdot e^{2Q(x)}a(0)v,\ \forall x\in U_0\text{ and }\quad\forall v\in\mathbb{R}^d; \tag{2.28}$$

$$\frac{1}{C_1}|v|^2\le v\cdot(\mu a)(x)v\le C_1|v|^2,\quad\forall x\in U_0\text{ and }\forall v\in\mathbb{R}^d; \tag{2.29}$$

$$\sup_{x\in U_0}|e^{2Q(x)}|+\sup_{x\in U_0}|\nabla e^{2Q(x)}|\le C_1. \tag{2.30}$$

For a domain F we denote the principal eigenvalue of $-\Delta$ on F with the Dirichlet boundary condition on ∂F by ρ_F. In the sequel, whenever $A_\epsilon\subset\subset F$, we will write $\rho_{F,\epsilon}$ meaning $\rho_{F\backslash\overline{A_\epsilon}}$. We recall that in this case, $\rho_{F,\epsilon}>\rho_F$ and that in general, $\rho_F \to \infty$, as F shrinks to a point. We denote the operator norm of a real $d \times d$ matrix acting on $\mathbb{R}^d$, equipped with the standard Euclidean norm by $|||\cdot|||$.

Let $a_1(x) = \mu(0)e^{2Q(x)}a(0)$. For $\delta > 0$, we choose a domain $U_\delta \subset U_0$ with $0 \in U_\delta$ satisfying:

(2.31)
$$1 - \delta \leq \frac{v \cdot (\mu a)(x) v}{v \cdot a_1(y) v} \leq 1 + \delta, \quad \forall x, y \in U_\delta,\ v \in \mathbb{R}^d \backslash \{0\};$$

(2.32)
$$\sup_{x \in U_\delta} |||a(x) - a(0)||| + \sup_{x \in U_\delta} |||(\mu a)(x) - (\mu a)(0)||| + \sup_{x \in U_\delta} |b(x) - b(0)| < \delta;$$

(2.33)
$$\frac{\rho_{U_\delta}}{2C_1^2} > 1.$$

If Λ is a continuous positive definite matrix-valued function and φ is a vector field on $U_\delta \backslash \overline{A_\epsilon}$, then define

$$\|\varphi\|_\Lambda^2 = \frac{1}{2} \int_{U_\delta \backslash \overline{A_\epsilon}} \varphi \cdot \Lambda \varphi dx.$$

This definition should not be confused with the norm on $\mathbb{R}^d$ defined in (1.5). Abusing notation, we will use $\|\cdot\|$ as short notation for $\|\cdot\|_{Id}$ as well as for the L^2 norm for a scalar function on $U_\delta \backslash \overline{A_\epsilon}$.

Let u_ϵ denote the capacitary potential of A_ϵ in U_δ with respect to $e^{2Q}a(0)$. Let v_ϵ denote the $\mathcal{E}_{\mathcal{L}}$-capacitary potential of A_ϵ in U_δ. Set $\psi_\epsilon = u_\epsilon - v_\epsilon$. Then, We have

$$(2.34) \qquad \begin{cases} \mathcal{L}_0 \psi_\epsilon = \frac{1}{2} \nabla \cdot ((a - a(0)) \nabla v_\epsilon) + (\hat{b} - \hat{b}(0)) \cdot \nabla v_\epsilon & \text{in } U_\delta \backslash \overline{A_\epsilon}; \\ \psi_\epsilon = 0 & \text{on } \partial U_\delta \cup \partial A_\epsilon. \end{cases}$$

Multiply both sides by $-e^{2Q}\psi_\epsilon$ and integrate over $U_\delta \backslash \overline{A_\epsilon}$. One obtains

(2.35)
$$\begin{aligned} &- \int_{U_\delta \backslash \overline{A_\epsilon}} \psi_\epsilon e^{2Q} \mathcal{L}_0 \psi_\epsilon dx \\ &\quad = -\frac{1}{2} \int_{U_\delta \backslash \overline{A_\epsilon}} \psi_\epsilon e^{2Q} \nabla \cdot ((a - a(0)) \nabla v_\epsilon) dx - \int_{U_\delta \backslash \overline{A_\epsilon}} \psi_\epsilon e^{2Q} (\hat{b} - \hat{b}(0)) \cdot \nabla v_\epsilon dx \\ &\quad \equiv (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

On the other hand, since ψ_ϵ vanishes on $\partial U_\delta \cup \partial A_\epsilon$, the divergence theorem gives

(2.36)
$$\|\nabla\psi_\epsilon\|^2_{e^{2Q}a(0)} = \frac{1}{2}\int_{U_\delta\setminus\overline{A_\epsilon}} \nabla\psi_\epsilon \cdot e^{2Q}a(0)\nabla\psi_\epsilon dx = -\int_{U_\delta\setminus\overline{A_\epsilon}} \psi_\epsilon e^{2Q}\mathcal{L}_0\psi_\epsilon dx.$$

Thus, by (2.28), (2.35) and (2.36),

$$\|\nabla\psi_\epsilon\|^2 \leq C_1\left[(\mathrm{I}) + (\mathrm{II})\right] \tag{2.37}$$

By (2.30) and (2.32),

$$|(\mathrm{II})| \leq \delta C_1\|\psi_\epsilon\|\|\nabla v_\epsilon\| \leq \delta C_1(\|\psi_\epsilon\| + \|\nabla\psi_\epsilon\|)\|\nabla v_\epsilon\|. \tag{2.38}$$

The divergence theorem gives

$$|(\mathrm{I})| = \left|\frac{1}{2}\int_{U_\delta\setminus\overline{A_\epsilon}} \nabla(e^{2Q}\psi_\epsilon)\cdot(a - a(0))\nabla v_\epsilon dx\right|.$$

By (2.32) we have

$$|(\mathrm{I})| \leq \frac{\delta}{2}\int_{U_\delta\setminus\overline{A_\epsilon}} |\nabla(e^{2Q}\psi)||\nabla v_\epsilon| dx. \tag{2.39}$$

Now, $|\nabla(e^{2Q}\psi_\epsilon)| \leq |\psi_\epsilon\nabla e^{2Q}| + |e^{2Q}\nabla\psi_\epsilon|$, therefore by (2.30), $|\nabla(e^{2Q}\psi_\epsilon)| \leq C_1(|\nabla\psi_\epsilon| + |\psi_\epsilon|)$ and it follows from (2.39) that

$$|(\mathrm{I})| \leq \delta C_1(\|\psi_\epsilon\| + \|\nabla\psi_\epsilon\|)\|\nabla v_\epsilon\|. \tag{2.40}$$

Thus from (2.37), (2.40) and (2.38) we obtain

$$\|\nabla\psi_\epsilon\|^2 \leq 2C_1^2\delta(\|\psi_\epsilon\| + \|\nabla\psi_\epsilon\|)\|\nabla v_\epsilon\|, \tag{2.41}$$

By the Poincaré inequality,

$$\rho_{U_\delta,\epsilon}\|\psi_\epsilon\|^2 \leq \|\nabla\psi_\epsilon\|^2.$$

Therefore (2.41) gives

$$\rho_{U_\delta,\epsilon}\|\psi_\epsilon\|^2 - 2C_1^2\delta\|\nabla v_\epsilon\|\|\psi_\epsilon\| - 2C_1^2\delta\|\nabla\psi_\epsilon\|\|\nabla v_\epsilon\| \leq 0.$$

This is a quadratic inequality in $\|\psi_\epsilon\|$. Thus,

$$\begin{aligned}\|\psi_\epsilon\| &\leq \frac{2C_1^2\delta\|\nabla v_\epsilon\| + \sqrt{4C_1^4\delta^2\|\nabla v_\epsilon\|^2 + 8C_1^2\delta\rho_{U_\delta,\epsilon}\|\nabla\psi_\epsilon\|\|\nabla v_\epsilon\|}}{2\rho_{U_\delta,\epsilon}} \\ &= \frac{C_1^2\delta\|\nabla v_\epsilon\|\left(1 + \sqrt{1 + \frac{2\rho_{U_\delta,\epsilon}\|\nabla\psi_\epsilon\|}{C_1^2\delta\|\nabla v_\epsilon\|}}\right)}{\rho_{U_\delta,\epsilon}}\end{aligned}$$

Plugging this inequality back in (2.41), we obtain

$$\|\nabla\psi_\epsilon\|^2 \leq \frac{2C_1^4\delta^2\|\nabla v_\epsilon\|^2}{\rho_{U_\delta,\epsilon}}\left(1+\sqrt{1+\frac{2\rho_{U_\delta,\epsilon}\|\nabla\psi_\epsilon\|}{C_1^2\delta\|\nabla v_\epsilon\|}}\right)+2C_1^2\delta\|\nabla\psi_\epsilon\|\|\nabla v_\epsilon\|.$$

After dividing by $(\delta\|\nabla v_\epsilon\|)^2$, we are left with the inequality

$$\left(\frac{\|\nabla\psi_\epsilon\|}{\delta\|\nabla v_\epsilon\|}\right)^2 \leq \frac{2C_1^4\left(1+\sqrt{1+\frac{2\rho_{U_\delta,\epsilon}}{C_1^2}\frac{\|\nabla\psi_\epsilon\|}{\delta\|\nabla v_\epsilon\|}}\right)}{\rho_{U_\delta,\epsilon}}+2C_1^2\frac{\|\nabla\psi_\epsilon\|}{\delta\|\nabla v_\epsilon\|}.$$

Letting $\zeta=\frac{\|\nabla\psi_\epsilon\|}{\delta\|\nabla v_\epsilon\|}$, the above inequality may be rewritten as

$$\zeta^2 \leq \frac{2C_1^4}{\rho_{U_\delta,\epsilon}}\left(1+\sqrt{1+\frac{2\rho_{U_\delta,\epsilon}}{C_1^2}\zeta}\right)+2C_1^2\zeta. \tag{2.42}$$

Assume that $\zeta>1$. It follows from (2.33) that the expression under the root sign in (2.42) is bounded above by $2\frac{2\rho_{U_\delta,\epsilon}\zeta}{C_1^2}<(\frac{2\rho_{U_\delta,\epsilon}\zeta}{C_1^2})^2$. We obtain

$$\zeta^2 \leq \frac{2C_1^4}{\rho_{U_\delta,\epsilon}}+\frac{2C_1^4}{\rho_{U_\delta,\epsilon}}\frac{2\rho_{U_\delta,\epsilon}\zeta}{C_1^2}+2C_1^2\zeta < C_1^2\zeta+4C_1^2\zeta+2C_1^2\zeta=7C_1^2\zeta.$$

Thus, $\zeta\leq C_3\equiv\max(7C_1^2,1)$. From the definition of ζ we obtain

$$\|\nabla\psi_\epsilon\|\leq\delta C_3\|\nabla v_\epsilon\|.$$

With (2.29) we have

$$\|\nabla\psi_\epsilon\|_{\mu a}\leq\sqrt{C_1}\|\nabla\psi_\epsilon\|\leq\delta\sqrt{C_1}C_3\|\nabla v_\epsilon\|. \tag{2.43}$$

Since

$$|\,\|\nabla u_\epsilon\|_{\mu a}-\|\nabla v_\epsilon\|_{\mu a}\,|\leq\|\nabla\psi_\epsilon\|_{\mu a},$$

we obtain

$$|\,\|\nabla u_\epsilon\|_{\mu a}-\|\nabla v_\epsilon\|_{\mu a}\,|\leq\delta\sqrt{C_1}C_3\|\nabla v_\epsilon\|,$$

where the second inequality follows from (2.43). Dividing by $\|\nabla v_\epsilon\|_{\mu a}$, we obtain

$$1-\delta\sqrt{C_1}C_3\frac{\|\nabla u_\epsilon\|_{\mu a}}{\|\nabla v_\epsilon\|_{\mu a}}\leq 1+\delta\sqrt{C_1}C_3.$$

Recall that $\|\nabla v_\epsilon\|^2_{\mu a}=\mathcal{E}_{\mathcal{L}}(A_\epsilon,U_\delta)$. Taking the squares of the above inequalities, we get

$$(1-\delta\sqrt{C_1}C_3)^2\frac{\|\nabla u_\epsilon\|^2_{\mu a}}{\mathcal{E}_{\mathcal{L}}(A_\epsilon,U_\delta)}\leq(1+\delta\sqrt{C_1}C_3)^2. \tag{2.44}$$

Finally, we compare between $\|\nabla u_\epsilon\|^2_{\mu a}$ and $\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon, U_\delta)$. Let w_ϵ denote the capacitary potential of A_ϵ in U_δ with respect to $a(0)$. Define the set M by letting

$$M = \{v \in C_c^\infty(U_\delta) : v \geq 1 \text{ on } A_\epsilon\}.$$

We have

$$\mathrm{Cap}_{a(0)}(A_\epsilon, U_\delta) = \|\nabla w_\epsilon\|^2_{a(0)} = \inf_{v\in M} \|\nabla v\|^2_{a(0)}$$

and

$$\mathrm{Cap}_{a_1}(A_\epsilon, U_\delta) = \|\nabla u_\epsilon\|^2_{a_1} = \inf_{v\in M} \|\nabla v\|^2_{a_1}.$$

By (2.31),

$$\begin{aligned}\|\nabla u_\epsilon\|^2_{\mu a} &\leq (1+\delta)\mathrm{Cap}_{a_1}(A_\epsilon, U_\delta) \leq (1+\delta)\|\nabla w_\epsilon\|^2_{a_1}\\ &\leq \frac{1+\delta}{1-\delta}\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon, U_\delta)\\ &= \frac{1+\delta}{1-\delta}\mu(0)\|\nabla w_\epsilon\|^2_{a(0)} \leq \frac{1+\delta}{1-\delta}\|\nabla u_\epsilon\|^2_{(\mu a)(0)}.\end{aligned}$$

It also follows from (2.31) that

$$\|\nabla u_\epsilon\|^2_{(\mu a)(0)} \leq \frac{1+\delta}{1-\delta}\|\nabla u_\epsilon\|^2_{\mu a}.$$

We, therefore, obtain

$$\frac{1-\delta}{1+\delta} \leq \frac{\|\nabla u_\epsilon\|^2_{\mu a}}{\mu(0)\mathrm{Cap}_{a(0)}(A_\epsilon, U_\delta)} \leq \frac{1+\delta}{1-\delta}. \tag{2.45}$$

Thus, (2.27) follows immediately from (2.44) and (2.45). ∎

3. Proof of Theorem 1.3

In order to exploit the radial symmetry the model has near the centers of the obstacles, we will assume that $U \subset D$ is the disjoint union of balls U_j, $j = 1, \ldots, n$, all of which with the same radius, the center of U_j being x_j. In what follows, we will only consider ϵ sufficiently small, such that $A^j_\epsilon \subset U_j$ for all j. This choice of U is in contrast with the hard obstacle case where U was taken to be connected. We let $m_{1,\epsilon}$ and $m_{2,\epsilon}$ denote the invariant measures on ∂A_ϵ and ∂U, respectively, as defined in the beginning of Section 2. Although U is not a domain unless $n = 1$, Lemma 2.8 remains true in this case as well, the proof requires no changes. As a result, we note that $m_{1,\epsilon}$ is uniform on ∂A^j_ϵ and $m_{1,\epsilon}(A^j_\epsilon) = 1/n$.

We state two lemmas and then give the proof of Theorem 1.3. After that, we return to prove the lemmas.

Lemma 3.1: *Let*

$$\rho_\epsilon \quad = \quad \mathbb{E}_{m_{2,\epsilon}} \exp\Big(- \int_0^{\sigma_2} W_\epsilon(X(s))ds \Big).$$

Then,

$$\mathbb{E}_{m_{2,\epsilon}} \int_0^\infty \exp\Big(- \int_0^t W_\epsilon(X(s))ds \Big) dt \sim \frac{\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon}{1-\rho_\epsilon}, \qquad \text{as } \epsilon \to 0.$$

Lemma 3.2:

$$-(\lambda_{c,\epsilon})^{-1} \sim \mathbb{E}_{m_{2,\epsilon}} \int_0^\infty \exp\Big(- \int_0^t W_\epsilon(X(s))ds \Big) dt.$$

Proof of Theorem 1.3. By Lemmas 3.1 and 3.2,

$$-\lambda_{c,\epsilon} \sim \frac{1-\rho_\epsilon}{\mathbb{E}_{m_{2,\epsilon}\tau_\epsilon}},$$

as $\epsilon \to 0$. The asymptotic behavior of $(\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon)^{-1}$ is already known from Proposition 2.1 and Theorem 1.2:

$$(\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon)^{-1} \sim \begin{cases} \frac{\pi}{\ln \epsilon^{-1}} \sum_{j=1}^n \widetilde{\phi_c}(x_j) & d=2; \\ \frac{d(d-2)\omega_d \epsilon^{d-2}}{2} \sum_{j=1}^n \widetilde{\phi_c}(x_j) & d \geq 3. \end{cases}$$

Thus,

$$-\lambda_{c,\epsilon} \sim (1-\rho_\epsilon) \times \begin{cases} \frac{\pi}{\ln \epsilon^{-1}} \sum_{j=1}^n \widetilde{\phi_c}(x_j) & d=2; \\ \frac{d(d-2)\omega_d \epsilon^{d-2}}{2} \sum_{j=1}^n \widetilde{\phi_c}(x_j) & d \geq 3. \end{cases} \tag{3.1}$$

It only remains to evaluate the asymptotics of $1-\rho_\epsilon$. Without loss of generality, we may assume that $x_1 = 0$. We note that $\rho_\epsilon = \mathbb{E}_{m_{1,\epsilon}} \exp(-\int_0^{\eta_1} W_\epsilon(X(s))ds)$. Let e_1 denote the vector $(1,0,\ldots,0) \in \mathbb{R}^d$. Due to radial symmetry, we have

$$\rho_\epsilon = \mathbb{E}_{\epsilon e_1} \exp\Big(- \int_0^{\eta_1} W_\epsilon(X(s))ds \Big) \tag{3.2}$$

It is easier to work with an obstacle whose support is fixed, rather than a shrinking one. Such a reduction is achieved by Brownian scaling. Let Y be Brownian motion on $\mathbb{R}^d$. For $E \subset \mathbb{R}^d$ we let $\tau_E^Y = \inf\{t \geq 0 : Y(t) \in \partial E\}$. Let $x \in B_1(0)$. The distribution of $\{\epsilon^{-1}X(t\epsilon^2) : 0 \leq t < \epsilon^{-2}\tau_{B_1(0)}\}$, starting at x coincides with the distribution of $\{Y(t) : 0 \leq t < \tau^Y_{B_{\epsilon^{-1}}(0)}\}$, starting at $\epsilon^{-1}x$.

Abusing notation, we denote the function $x \to W(|x|)$ by W. After rescaling (3.2), ρ_ϵ has the following representation:

$$\rho_\epsilon = \mathbb{E}_{e_1} \exp\bigg(- g(\epsilon)\epsilon^2 \int_0^{\tau^Y_{B_{\epsilon^{-1}}(0)}} W(Y(s))ds \bigg).$$

The proof will be divided according to the dimension d and the asymptotics of $g(\epsilon)\epsilon^2$.

(1) $d \geq 3$ and $\gamma \in (0, \infty)$. Since Y is transient, bounded convergence gives

$$\rho \equiv \lim_{\epsilon \to 0} \rho_\epsilon = \mathbb{E}_{e_1} \exp\bigg(- \gamma \int_0^\infty W(Y(s))ds \bigg).$$

Hence $\rho = u(1)$, where $u = u(r)$ is the minimal positive solution to

$$\begin{cases} \frac{1}{2}u'' + \frac{d-1}{2r}u' - \gamma W u = 0 \quad \text{on } [0, \infty); \\ u'(0) = 0; \\ \lim_{r \to \infty} u(r) = 1. \end{cases} \tag{3.3}$$

Recall that when $\gamma \in (0, \infty)$, we are assuming that $W \equiv \beta \mathbf{1}_{[0,1]}$. We now show that in this case, ρ can be expressed in terms of modified Bessel functions. For $r \geq 1$, the differential equation in (3.3) is equivalent to $\frac{1}{2}(r^{1-d}(r^{d-1}u')') = 0$. Therefore,

$$u(r) = C_1 r^{2-d} + 1, \quad \text{for } r \geq 1.$$

For $r \in (0, 1)$, we transform the differential equation in (3.3) to a Bessel equation by a change of variables. Then u satisfies

$$u'' + \frac{d-1}{r}u' - 2\gamma\beta u = 0, \quad r \in (0, 1) \tag{3.4}$$

Let

$$w(z) = \bigg(\frac{z}{\sqrt{2\gamma\beta}} \bigg)^{d/2-1} u\Big(\frac{z}{\sqrt{2\gamma\beta}} \Big).$$

Then one easily verifies that

$$z^2 w'' + z w' - (z^2 + (1 - d/2)^2)w = 0, \quad z \in (0, \sqrt{2\gamma\beta}). \tag{3.5}$$

Equation (3.5) is a modified Bessel equation. Every solution is a linear combination of $I_{1-d/2}$ and $I_{d/2-1}$, where I_p is as in (1.6) [11, page 138]. Since $u(z) = z^{1-d/2} w(\sqrt{2\gamma\beta} z)$, it follows that

$$u(z) = z^{1-d/2} \Big(A I_{d/2-1}(\sqrt{2\gamma\beta} z) + B I_{1-d/2}(\sqrt{2\gamma\beta} z) \Big). \tag{3.6}$$

It follows from the power series representation (1.6) that

$$\lim_{z\to 0} |z^{1-d/2} I_{1-d/2}(\sqrt{2\gamma\beta}z)| = \infty$$

and that $z^{1-d/2} I_{d/2-1}(\sqrt{2\gamma\beta}z)$ is analytic with the exception of a removable singularity at the origin. These observations imply that $B = 0$. One can verify directly from (1.6) that

$$(z^{-p} I_p)'(z) = z^{-p} I_{p+1}(z).$$

Therefore, $u'(1^-) = \sqrt{2\gamma\beta} A I_{d/2}(\sqrt{2\gamma\beta})$. Since u and u' are continuous at 1, C_1 and A are determined by the following equations:

$$\begin{aligned} C_1 + 1 &= u(1^+) = u(1^-) = A I_{d/2-1}(\sqrt{2\gamma\beta}) \\ (2-d)C_1 &= u'(1^+) = u'(1^-) = A\sqrt{2\gamma\beta} I_{d/2}(\sqrt{2\gamma\beta}). \end{aligned}$$

Since $\rho = u(1) = C_1 + 1$, after solving these equations we find that

$$1 - \rho = -C_1 = \frac{\sqrt{2\gamma\beta} I_{d/2}(\sqrt{2\gamma\beta})}{(d-2) I_{d/2-1}(\sqrt{2\gamma\beta}) + \sqrt{2\gamma\beta} I_{d/2}(\sqrt{2\gamma\beta})}.$$

This with (3.1) gives Theorem 1.3 (2) for $d \geq 3$.

(2) $d \geq 3$ and $\gamma = \infty$. By bounded convergence, $\lim_{\epsilon\to 0} \rho_\epsilon = 0$; hence $1 - \rho_\epsilon \sim 1$. This with (3.1) gives Theorem 1.3 (3) for $d \geq 3$.

(3) $d = 2$ and $\gamma \in (0, \infty)$. The proof is essentially the same as for the corresponding case where $d \geq 3$. However, because of recurrence, we cannot take the limit $\epsilon \to 0$. Therefore, (3.3) is replaced by the following problem:

$$\begin{cases} \frac{1}{2}u'' + \frac{1}{2r}u' - g(\epsilon)\epsilon^2 \mathbf{1}_{[0,1]} u = 0 & r \in (0, \epsilon^{-1}); \\ u'(0) = 0; \\ u(\epsilon^{-1}) = 1; \end{cases} \tag{3.7}$$

We denote the solution by u_ϵ. Then clearly, $\rho_\epsilon = u_\epsilon(1)$. We have

$$u_\epsilon(r) = C_1(\ln r + \ln \epsilon) + 1, \text{ for } r \in (1, \epsilon^{-1}).$$

The discussion following (3.3) leading to (3.6) was independent of the assumption on the dimension, therefore (3.6) also holds in the present case. In particular,

$$u_\epsilon(r) = A I_0(\sqrt{2g(\epsilon)}\epsilon r), \quad r \leq \epsilon^{-1}.$$

The compatibility equations are the following:

$$C_1 \ln\epsilon + 1 = u_\epsilon(1^+) = u_\epsilon(1^-) = AI_0(\sqrt{2g(\epsilon)}\epsilon)$$
$$C_1 = u'_\epsilon(1^+) = u'_\epsilon(1^-) = A\sqrt{2g(\epsilon)}\epsilon I_1(\sqrt{2g(\epsilon)}\epsilon),$$

From which we obtain

$$(1-\rho_\epsilon) = (1-u_\epsilon(1)) = \Big(1 + \frac{I_0(\sqrt{2g(\epsilon)}\epsilon)}{\sqrt{2g(\epsilon)}\epsilon \ln\epsilon^{-1} I_1(\sqrt{2g(\epsilon)}\epsilon)}\Big)^{-1}. \tag{3.8}$$

By (1.6), $I_0(z) = \sum_{k=0}^\infty \frac{(z/2)^{2k}}{(k!)^2}$ and $I_1(z) = \sum_{k=0}^\infty \frac{(z/2)^{1+2k}}{k!(k+1)!}$. In particular, $I_0(z)/I_1(z) \sim 2/z$ as $z \to 0$. Since $\lim_{\epsilon\to 0}\sqrt{2g(\epsilon)}\epsilon = 0$, we obtain

$$\frac{I_0(\sqrt{2g(\epsilon)}\epsilon)}{I_1(\sqrt{2g(\epsilon)}\epsilon)} \sim \frac{2}{\sqrt{2g(\epsilon)}\epsilon}.$$

Thus, by (3.8),

$$(1-\rho_\epsilon) \sim \Big(1 + \frac{1}{g(\epsilon)\epsilon^2\ln\epsilon^{-1}}\Big)^{-1} \sim \Big(1+\frac{1}{\gamma}\Big)^{-1}.$$

This with (3.1) gives Theorem 1.3-(2) for $d = 2$.

(4) $d = 2$ and $\gamma = \infty$. For every $\widetilde{\gamma} \in (0,\infty)$,

$$(1+1/\widetilde{\gamma})^{-1} \le \liminf_{\epsilon\to 0}(1-\rho_\epsilon) \le \limsup_{\epsilon\to 0}(1-\rho_\epsilon) \le 1.$$

Thus, we have

$$\lim_{\epsilon\to 0}(1-\rho_\epsilon) = 1.$$

This with (3.1) gives Theorem 1.3 (3) for $d = 2$.

(5) $d \ge 2$ and $\gamma = 0$. Define

$$u_\epsilon(x) = \mathbb{E}_x \exp\Big(- g(\epsilon)\epsilon^2 \int_0^{\tau^Y_{B_{\epsilon^{-1}}(0)}} W(Y(s))ds\Big), \quad x \in B_{\epsilon^{-1}}(0).$$

Then $\rho_\epsilon = u_\epsilon(e_1)$ and u_ϵ is the solution to

$$\begin{cases} \frac{1}{2}\Delta u - g(\epsilon)\epsilon^2 W u = 0 & \text{in } B_{\epsilon^{-1}}(0); \\ u = 1 & \text{on } \partial B_{\epsilon^{-1}}(0). \end{cases}$$

We also define

$$w_\epsilon(x) = g(\epsilon)\epsilon^2\mathbb{E}_x \int_0^{\tau^Y_{B_{\epsilon^{-1}}(0)}} W(Y(s))ds, \quad x \in B_{\epsilon^{-1}}(0) \text{ and}$$
$$v_\epsilon(x) = 1 - (1-w_\epsilon(0))w_\epsilon(x), \quad x \in B_{\epsilon^{-1}}(0).$$

Since $e^{-t} \geq 1-t,\ t \in \mathbb{R}$, we note that

$$1 - w_\epsilon(x) \leq u_\epsilon(x). \tag{3.9}$$

Consequently,

$$1 - \rho_\epsilon = 1 - u_\epsilon(e_1) \leq w_\epsilon(e_1). \tag{3.10}$$

It is clear that w_ϵ is radially symmetric and satisfies $\frac{1}{2}\Delta w_\epsilon = -g(\epsilon)\epsilon^2 W$. Therefore by solving the corresponding ODE's we can see that for all $|x| \leq 1$,

$$w_\epsilon(x) = \begin{cases} 2g(\epsilon)\epsilon^2 \int_0^1 W(t)t\left[\ln \epsilon^{-1} - \ln(|x| \vee t)\right] dt & d = 2; \\ \frac{2}{d-2} g(\epsilon)\epsilon^2 \int_0^1 W(t)t^{d-1}\left[(|x| \vee t)^{2-d} - \epsilon^{d-2}\right] dt & d \geq 3. \end{cases} \tag{3.11}$$

We observe that the maximum of w_ϵ is attained at 0. We claim that

$$\lim_{\epsilon \to 0} w_\epsilon(0) = 0. \tag{3.12}$$

If $d \geq 3$, then (3.12) holds as an immediate consequence of the definition of w_ϵ, because Y is transient. If $d = 2$, then by (3.11), $w_\epsilon(0) \sim 2g(\epsilon)\epsilon^2 \ln \epsilon^{-1} \int_0^1 W(t)tdt$. The last quantity goes to 0, as $\epsilon \to 0$, giving (3.12).

From now on we restrict the discussion to $\epsilon > 0$ sufficiently small so that $w_\epsilon(0) < 1$. Now,

$$\frac{1}{2}\Delta v_\epsilon = g(\epsilon)\epsilon^2(1 - w_\epsilon(0))W \leq g(\epsilon)\epsilon^2(1 - w_\epsilon(x))W \leq g(\epsilon)\epsilon^2 W u_\epsilon = \frac{1}{2}\Delta u_\epsilon,$$

where the last inequality follows from (3.9). Hence by the maximum principle, $v_\epsilon \geq u_\epsilon$. In particular,

$$1 - \rho_\epsilon = 1 - u_\epsilon(e_1) \geq 1 - v_\epsilon(e_1) = (1 - w_\epsilon(0))w_\epsilon(e_1). \tag{3.13}$$

Summarizing, the inequalities (3.13) and (3.10), give us

$$(1 - w_\epsilon(0))w_\epsilon(e_1) \leq 1 - \rho_\epsilon \leq w_\epsilon(e_1),$$

and it follows from (3.12) that

$$1 - \rho_\epsilon \sim w_\epsilon(e_1). \tag{3.14}$$

We continue according to the dimension d. If $d \geq 3$, then by (3.11) and (3.14) we have

$$1 - \rho_\epsilon \sim \frac{2g(\epsilon)\epsilon^2}{d-2} \int_0^1 W(t)t^{d-1}dt. \tag{3.15}$$

If $d = 2$, then by (3.11) and (3.14) we have

$$1 - \rho_\epsilon \sim 2g(\epsilon)\epsilon^2 \ln \epsilon^{-1} \int_0^1 W(t)tdt. \tag{3.16}$$

Theorem 1.3 (1) follows from (3.1), (3.15) and (3.16). ∎

Proof of Lemma 3.1. Decompose the time interval $[0,\infty)$ into cycles $[\sigma_n, \sigma_{n+1})$, where $n \in \mathbb{N}$. Each cycle consists of two parts — one part from the beginning of the cycle until the hitting of the obstacle, and the other part from the hitting of the obstacle until the end of the cycle. These two parts correspond, respectively, to the time intervals $[\sigma_n, \eta_n)$ and $[\eta_n, \sigma_{n+1})$. We have

$$\mathbb{E}_{m_{2,\epsilon}} \int_0^\infty \exp\left(- \int_0^t W_\epsilon(X(s))ds \right) dt = \sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}}(F_n + S_n),$$

where

$$F_n = \int_{\sigma_n}^{\eta_n} \exp\left(- \int_0^t W_\epsilon(X(s))ds \right) dt, and$$
$$S_n = \int_{\eta_n}^{\sigma_{n+1}} \exp\left(- \int_0^t W_\epsilon(X(s))ds \right) dt.$$

Note that in the first part, the process does not visit the obstacle, which implies that the integrand in the definition F_n is $\exp(-\int_0^{\sigma_n} W_\epsilon(X(s))ds)$. Thus,

$$F_n = (\eta_n - \sigma_n) \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right), \quad \text{and}$$
$$S_n = \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right) \int_{\eta_n}^{\sigma_{n+1}} \exp\left(- \int_{\eta_n}^t W_\epsilon(X(s))ds \right) dt. \tag{3.17}$$

We first estimate $\sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} F_n$. By the strong Markov property,

$$\mathbb{E}_{m_{2,\epsilon}} F_n = \mathbb{E}_{m_{2,\epsilon}} \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right) \mathbb{E}_{X(\sigma_n)} \tau_\epsilon.$$

By Corollary 2, for any $\delta \in (0,1)$,

$$(1-\delta)\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon \le \mathbb{E}_x \tau_\epsilon \le (1+\delta)\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon, \quad x \in \partial U,$$

provided that $\epsilon > 0$ is sufficiently small. Consequently,

$$\sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} F_n \sim \mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon \sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right), \quad \text{as } \epsilon \to 0.$$

We will show that

$$\mathbb{E}_{m_{2,\epsilon}} \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right) = \rho_\epsilon^{n-1}. \tag{3.18}$$

This yields

$$\sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} F_n \sim \frac{\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon}{1-\rho_\epsilon}, \qquad \text{as } \epsilon \to 0. \tag{3.19}$$

To prove (3.18), fix $n \in \mathbb{N}$ and note that

$$\mathbb{E}_{m_{2,\epsilon}} \exp\left(- \int_0^{\sigma_{n+1}} W_\epsilon(X(s))ds \right) = \mathbb{E}_{m_{1,\epsilon}} \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right).$$

By the strong Markov property,

$$\begin{aligned}\mathbb{E}_{m_{1,\epsilon}} &\left[\exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s)ds \right) | \mathcal{F}_{\eta_{n-1}} \right] \\ &= \exp\left(- \int_0^{\sigma_{n-1}} W_\epsilon(X(s))ds \right) \mathbb{E}_{X(\eta_{n-1})} \exp\left(- \int_0^{\sigma_1} W_\epsilon(X(s))ds \right).\end{aligned}$$

Due to radial symmetry, for $x \in \partial A_\epsilon$

$$\mathbb{E}_x \exp(- \int_0^{\sigma_1} W_\epsilon(X(s))ds) = \mathbb{E}_{m_{1,\epsilon}} \exp\left(- \int_0^{\sigma_1} W_\epsilon(X(s))ds \right) = \rho_\epsilon.$$

This implies that

$$\mathbb{E}_{m_{1,\epsilon}} \exp\left(- \int_0^{\sigma_n} W_\epsilon(X(s))ds \right) = \rho_\epsilon \mathbb{E}_{m_{1,\epsilon}} \exp\left(- \int_0^{\sigma_{n-1}} W_\epsilon(X(s))ds \right),$$

and (3.18) follows by induction.

We now estimate $\sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} S_n$. Let

$$R_\epsilon = \mathbb{E}_{m_{1,\epsilon}} \int_0^{\sigma_1} \exp\left(- \int_0^t W_\epsilon(X(s))ds \right) dt.$$

Again, by radial symmetry, for all $x \in \partial A_\epsilon$,

$$R_\epsilon = \mathbb{E}_x \int_0^{\sigma_1} \exp\left(- \int_0^t W_\epsilon(X(s))ds \right) dt. \tag{3.20}$$

We now prove that

$$\sum_{n=1}^\infty \mathbb{E}_{m_{2,\epsilon}} S_n = \frac{R_\epsilon}{1-\rho_\epsilon} = \frac{O(1)}{1-\rho_\epsilon}. \tag{3.21}$$

By (3.17) and the strong Markov property,

$$\begin{aligned}\mathbb{E}_{m_{2,\epsilon}} S_n = \mathbb{E}_{m_{2,\epsilon}} \Big[\exp\Big(-\int_0^{\sigma_n} W_\epsilon(X(s))ds\Big) \mathbb{E}_{X(\eta_n)} \\ \times \int_0^{\sigma_1} \exp\Big(-\int_0^t W_\epsilon(X(s))ds\Big) dt\Big] \\ = \mathbb{E}_{m_{2,\epsilon}} \exp\Big(-\int_0^{\sigma_n} W_\epsilon(X(s))ds\Big) R_\epsilon.\end{aligned}$$

Therefore, (3.18) and (3.20) imply that

$$\mathbb{E}_{m_{2,\epsilon}} S_n = \rho_\epsilon^{n-1} R_\epsilon.$$

Since $R_\epsilon = O(1)$, (3.21) follows. The lemma is an immediate consequence of (3.19) and (3.21). ∎

Proof of Lemma 3.2. We adopt the notation from Lemma 2.7. In addition, we let $G_\epsilon(\cdot,\cdot)$ denote the Green's function for $\mathcal{L}_\epsilon$. Abusing notation, the measure whose density is $\int G_\epsilon(x,\cdot)dm_{2,\epsilon}(x)$ will be also denoted by G_ϵ. By the Feynman-Kac formula

$$G_\epsilon(D) = \mathbb{E}_{m_{2,\epsilon}} \int_0^\infty \exp\Big(-\int_0^t W_\epsilon(X(s))ds\Big) dt.$$

Then by Lemma 3.1,

$$G_\epsilon(D) \sim \frac{\mathbb{E}_{m_{2,\epsilon}} \tau_\epsilon}{1-\rho_\epsilon}, \qquad \text{as } \epsilon \to 0. \tag{3.22}$$

Since

$$\phi_\epsilon(x) = -\lambda_{c,\epsilon} \int G_\epsilon(x,y)\phi_\epsilon(y)dy,$$

integrating both sides with respect to $dm_{2,\epsilon}$, the normalization $\int_{\partial U} \phi_\epsilon dm_{2,\epsilon} = 1$ (2.16) implies that

$$-\lambda_{c,\epsilon} = \Big(\int_D \phi_\epsilon dG_\epsilon \Big)^{-1}. \tag{3.23}$$

In light of (3.22) and (3.23), in order to prove the lemma, it is enough to show that

$$\int_D \phi_\epsilon dG_\epsilon \sim G_\epsilon(D), \qquad \text{as } \epsilon \to 0. \tag{3.24}$$

It is clear that

$$G_\epsilon(D) - \int_D |1-\phi_\epsilon| dG_\epsilon \le \int_D \phi_\epsilon dG_\epsilon \le G_\epsilon(D) + \int_D |1-\phi_\epsilon| dG_\epsilon.$$

Hence (3.24) will be established once we show that

$$\lim_{\epsilon\to 0}\frac{\int_D |1-\phi_\epsilon|dG_\epsilon}{G_\epsilon(D)}=0. \tag{3.25}$$

We now prove (3.25). Fix $\delta>0$. Let $V\subset\subset D\backslash\{x_1,\dots,x_n\}$ be closed and let $E=D\backslash V$. We further assume that $\mu(E)<\delta$. By Lemma 2.7, $\{\phi_\epsilon\}_{\epsilon>0}$ converges uniformly to 1 on V. By recurrence, $\lim_{\epsilon\to 0}G_\epsilon(D)=\infty$, and

$$\limsup_{\epsilon\to 0}\frac{\int_V |1-\phi_\epsilon|dG_\epsilon}{G_\epsilon(D)}=0. \tag{3.26}$$

We will prove below that

$$\limsup_{\epsilon\to 0}\frac{G_\epsilon(E)}{G_\epsilon(D)}\le 2\delta. \tag{3.27}$$

Lemma 2.7 also guarantees that $\{\phi_\epsilon\}_{\epsilon>0}$ are uniformly bounded, say by $C>0$. It follows from (3.27) that

$$\int_E |1-\phi_\epsilon|dG_\epsilon<(1+C)G_\epsilon(E)<3\delta(1+C)G_\epsilon(D),\quad \text{for } \epsilon \text{ sufficiently small.}$$

Using this with (3.26) gives

$$\limsup_{\epsilon\to 0}\frac{\int_D |1-\phi_\epsilon|dG_\epsilon}{G_\epsilon(D)}<3\delta(1+C).$$

Since δ is arbitrary, this proves (3.25). It remains to prove (3.27). Similarly to what we did in the proof of Lemma 3.1, we can write

$$G_\epsilon(E)=\sum_{n=1}^\infty \overline{F}_n+\sum_{n=1}^\infty \overline{S}_n,$$

where

$$\overline{S}_n=\mathbb{E}_{m_{2,\epsilon}}\int_{\sigma_n}^{\eta_n}\mathbf{1}_E(X(t))\exp\Big(-\int_0^t W_\epsilon(X(s))ds\Big)dt,\ \text{and}$$
$$\overline{F}_n=\mathbb{E}_{m_{2,\epsilon}}\int_{\eta_n}^{\sigma_{n+1}}\mathbf{1}_E(X(t))\exp\Big(-\int_0^t W_\epsilon(X(s))ds\Big)dt.$$

Let f be a smooth function such that $f\ge\mathbf{1}_E$. Since we assume that $\mu(E)<\delta$, we may choose f that also satisfies $\int_D f\,d\mu\le\delta$. We have

$$\begin{aligned}\overline{F}_n&=\mathbb{E}_{m_{2,\epsilon}}\exp\Big(-\int_0^{\sigma_n}W_\epsilon(X(s))ds\Big)\mathbb{E}_{X(\sigma_n)}\int_0^{\eta_1}\mathbf{1}_E(X(t))dt\\&\le\mathbb{E}_{m_{2,\epsilon}}\exp\Big(-\int_0^{\sigma_n}W_\epsilon(X(s))ds\Big)\mathbb{E}_{X(\sigma_n)}\int_0^{\eta_1}f(X(t))dt.\end{aligned}$$

Therefore by (3.18),

$$\overline{F}_n \le \rho_\epsilon^{n-1} \max_{x\in\partial U} \mathbb{E}_x \int_0^{\tau_\epsilon} f(X(t))dt.$$

By Lemma 2.6, we then obtain

$$\overline{F}_n \le 2\rho_\epsilon^{n-1}\mathbb{E}_{m_{2,\epsilon}} \int_0^{\tau_\epsilon} f(X(t))dt,$$

provided that ϵ is sufficiently small. Next, the last term on the right-hand side is bounded above by $\mathbb{E}_{m_{2,\epsilon}} \int_0^{\sigma_2} f(X(t))dt = \mathbb{E}_{m_{1,\epsilon}} \int_0^{\eta_1} f(X(t))dt$. By Khasminskii's construction in (2.3), the last quantity is equal to

$$\int_D f d\mu \mathbb{E}_{m_{1,\epsilon}}\eta_1 = \int f d\mu (O(1) + \mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon).$$

It follows that

$$\overline{F}_n \le 2\rho_\epsilon^{n-1}\delta(\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon + O(1)).$$

Thus,

$$\sum_{n=1}^{\infty} \overline{F}_n \le \frac{2\delta(\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon + O(1))}{1-\rho_\epsilon}. \tag{3.28}$$

Note that

$$\overline{S}_n \le \mathbb{E}_{m_{2,\epsilon}} S_n,$$

where S_n is as in the proof of Lemma 3.1. Hence by (3.28) and (3.21) we get

$$G_\epsilon(E) = \sum_{n=1}^{\infty} (\overline{F}_n + \overline{S}_n) \le \frac{2\delta(\mathbb{E}_{m_{2,\epsilon}}\tau_\epsilon + O(1)) + O(1)}{1-\rho_\epsilon}. \tag{3.29}$$

Finally, (3.27) follows from (3.29) and (3.22). ∎

4. Proof of Theorem 1.4

We will prove the theorem for $d \ge 3$, the case $d = 2$ being treated **mutatis mutandi**.

Proof. Since $(-\lambda_c)$ is obviously nondecreasing in β, it is sufficient to prove the theorem for $\beta \in [0,\infty)$. By hypothesis,

$$\lim_{m\to\infty} mr(m)^{d-2} = \beta \in [0,\infty), \quad \text{and}$$
$$m\delta(m)^d \sim |D|.$$

Therefore,

$$\lim_{m\to\infty} \delta(m)^{-d} r(m)^{d-2} = \beta/|D|. \tag{4.1}$$

Since $\lim_{m\to\infty} r(m) = 0$, this implies that

$$\lim_{m\to\infty} r(m)/\delta(m) = 0. \tag{4.2}$$

Let $\lambda(r,\delta)$ denote the principal eigenvalue for $\frac{1}{2}\Delta$ in the box $[-\delta/2,\delta/2]^d$ subject to the obstacle rK. By scaling, we see that $\lambda(r,\delta) = \delta^{-2}\lambda(r/\delta,1)$. It follows from (4.2) and Theorem 1.1 that

$$-\lambda(r(m),\delta(m)) \sim \frac{r(m)^{d-2}}{\delta(m)^d}\mathrm{Cap}_{Id}(K).$$

Therefore (4.1) gives

$$\lim_{m\to\infty} \lambda(r(m),\delta(m)) = -\frac{\beta\mathrm{Cap}_{Id}(K)}{|D|}. \tag{4.3}$$

We extend the obstacle A_m to $\mathbb{R}^d$, by translations of the $\delta(m)$-lattice. Let C_m be the maximal union of $\delta(m)$-translations of the hypercube $[-\frac{\delta(m)}{2},\frac{\delta(m)}{2}]^d$ which are contained in D. Let B_m be the minimal union of translations of such hypercubes which contains D. By symmetry, the principal eigenvalue for a single hypercube with the Neumann boundary condition on its boundary coincides with that of C_m and with that of B_m. It is also clear that if φ_m is the corresponding eigenfunction for B_m, then in fact it is periodic with respect to the lattice. By the Rayleigh-Ritz formula, $-\lambda_c^{(m)} = \inf_{u\in C^\infty(D)} \frac{\int_D |\nabla u|^2 dx}{\int_D u^2 dx}$. It follows that $-\lambda_c^{(m)} \le \frac{\int_D |\nabla\varphi_m|^2 dx}{\int_D \varphi_m^2 dx}$. The numerator is less than or equal to $\int_{B_m} |\nabla\varphi_m|^2 dx$. If $L(m)$ is the number of hypercubes in B_m, then $M(m)/L(m) \sim 1$. Now $\int_D \varphi_m^2 \ge \int_{C_m}\varphi_m^2 = \frac{M(m)}{L(m)}\int_{B_m}\varphi_m^2$. Letting $m\to\infty$, we obtain

$$\limsup_{m\to\infty}(-\lambda_c^{(m)}) \le \lim_{m\to\infty}(-\lambda(r(m),\delta(m))). \tag{4.4}$$

The lower bound requires more work. In what follows, C is a positive constant which may change from line to line. Let u_m be the eigenfunction for corresponding to $\lambda_c^{(m)}$, normalized by $\int_D u_m^2 = 1$. For $u \in W^{1,2}(D)$ let $\|u\|_{2,1} = \sqrt{\||\nabla u|\|_2^2 + \|u\|_2^2}$ be its Sobolev space norm. Then $\|u_m\|_{2,1} = \sqrt{1-\lambda_c^{(m)}}$. Thus by (4.4), $\|u_m\|_{2,1} \le C$ for all $m\in\mathbb{N}$. By the Sobolev inequality, $\|u_m\|_q \le C\|u_m\|_{2,1}$, where q satisfies $1/q = 1/2 - 1/d$. Thus, $\|u_m\|_q \le C$.

Let μ denote Lebesgue measure on $\mathbb{R}^d$. Then for every measurable $A \subset \overline{D}$, Hölder's inequality gives

$$\int_A u_m^2 \leq \left(\int u_m^q \right)^{2/q} \mu(A)^{(q-2)/q} \leq C\mu(A)^{(q-2)/q}. \tag{4.5}$$

We can now prove the lower bound. By the Rayleigh–Ritz formula,

$$-\lambda(r(m), \delta(m)) = \inf_{u \in C^\infty(C_m)} \frac{\int_{C_m} |\nabla u|^2}{\int_{C_m} u^2} \leq \frac{\int_{C_m} |\nabla u_m|^2}{\int_{C_m} u_m^2}.$$

The numerator on the right-hand side is bounded above by $\int_D |\nabla u_m|^2$. By (4.5), the denominator is bounded below by $1 - C\mu(D \backslash C_m)^{\frac{q-2}{q}}$. Hence

$$-\lambda(r(m), \delta(m)) \leq (1 - C\mu(D \backslash C_m)^{\frac{q-2}{2}})^{-1}(-\lambda_c^{(m)}),$$

and we obtain the lower bound

$$\liminf_{m \to \infty} (-\lambda_c^{(m)}) \geq \lim_{m \to \infty} -\lambda(r(m), \delta(m)) \tag{4.6}$$

The theorem now follows from (4.3), (4.4) and (4.6). ∎

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